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Fitting Creep-Rupture Life Distribution Using Accelerated Life Testing Data

This paper comments on using the Larson-Miller parameter to fit the creep-rupture life distribution as a function of temperature and stress. The commonly used least-squares linear regression method assumes that the creep-rupture life follows the lognormal distribution. Most engineering literature does not discuss the validity of this assumption. In this paper, we outline the procedure for validating two critical assumptions when the least-squares method is used. The maximum likelihood method is suggested as an alternative and more powerful method for fitting creep-rupture life distributions. Examples are given to demonstrate the use of these two methods using Microsoft Excel and the LIFEREG procedure in SAS. [S0094-9930(00)00504-7]

Keywords: Creep, Larson-Miller Parameter, Data Analysis, Life Distribution, Least-Squares Method, Maximum Likelihood Method

1 Introduction

Manufacturers usually conduct accelerated creep-rupture life tests of materials. The data obtained from these tests include temperature, stress, and time to rupture. The data from these tests are used to estimate the life distribution of these materials. Various equations have been proposed to represent the relationships among these variables [1]. The use of time-temperature parameters (TTPs) for presenting and extrapolating high-temperature creep-rupture data has been practiced for many years. A TTP is basically a function correlating the creep-rupture test variables of stress (load), temperature, and time.

One of the commonly used time-temperature parameters is the Larson-Miller (L-M) parameter

$$\mathrm{LMP}=f(S)=(T_c+273.2)(\log t_r+C) \tag{1}$$

where LMP is the Larson-Miller parameter, S is the stress, T_c is temperature in Celsius, t_r is time to rupture in hours, log represents the common (base 10) logarithm, and C is a material constant [2]. The stress may be measured in N/m^2 or psi. In this paper, we use kilo pounds per square inch (ksi) as the unit of measure for stress.

As to the form of $f(S)$ in Eq. (1), Jaske and Simonen [3] used the following equation for the HP-50 and Nb-modified HP alloys:

$$\mathrm{LMP}=f(S)=C_1+C_2\log S+C_3(\log S)^2 \tag{2}$$

where C_1, C_2, and C_3 are material constants to be determined from analysis of accelerated creep-rupture life testing data. Combining Eqs. (1) and (2), we obtain the following equation:

$$\log t_r=-C+\frac{C_1}{T_c+273.2}+C_2\frac{\log S}{T_c+273.2}+C_3\frac{(\log S)^2}{T_c+273.2} \tag{3}$$

Jaske and Simonen [3] used Eq. (3) and the least-squares regression technique to determine these material constants. Equations (1) and (2) were then used to calculate the average life at given temperature and stress. For predicting the time to the first failure of a fired tube in a furnace, they applied a reduction factor of five to the average life to compute a minimum life. Jaske and Simonen [3] also compared their approach to the one suggested by API RP-530 [4] and found a significant difference between the estimated minimum lives by the two approaches. The approach used by Jaske and Simonen [3] was more accurate because API-530 used the simple mean diameter formula to compute a single stress value, while the method by Jaske and Simonen [3] used finite element stress analysis to compute the stress distribution through the tube wall as a function of operating history and considered both creep and the relaxation and redistribution of thermal stress.

Some material manufacturers use a different mathematical equation to represent the relationship between LMP and $\log S$, as shown in the following:

$$\log S=A_1+A_2\times\mathrm{LMP}+A_3\times(\mathrm{LMP})^2 \tag{4}$$

where A_1, A_2, and A_3, are material constants to be determined from analysis of accelerated life testing data. These manufacturers also use the least-squares method to determine these material constants. They obtain the lower 95-percent confidence limit of $\log S$ for the purpose of calculating the minimum creep-rupture life of the alloy.

Nelson [5] mentions that the techniques for accelerated life data analysis can be used to estimate various life percentiles of components using the Larson-Miller parameter. He also states that the scatter in life is often ignored in metallurgical studies. Neither the manufacturers nor Jaske and Simonen [3] address the statistical distributions that the creep-rupture life may follow.

Because the least-squares regression method was used by Jaske and Simonen [3] in finding the constants in Eq. (3), the underlying assumption was that $\log t_r$ was a random variable following the normal distribution. However, the authors did not discuss the validity of this assumption. The $\log t_r$ may very well follow some other distributions, such as the Weibull distribution.

In the mathematical model represented by Eq. (4), $\log S$ was treated as a dependent variable, while LMP was treated as an independent variable. It assumed that most of the variations in $\log S$ could be represented by a second-order polynomial function of LMP and the residual variations in $\log S$ followed a standard normal distribution. These assumptions contradict the ones used by Jaske and Simonen [3]. Among temperature, stress, and creep-rupture life, both temperature and stress can be controlled via heat source and pressure, respectively. Creep-rupture life should be treated as a dependent variable following some statistical distribution. Thus, using Eq. (4) to estimate the remaining life would not give correct results.

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To find the mathematical relationship between LMP and stress S, most researchers use the least-squares regression technique. Conway [6] provided a detailed description on how the least-squares method should be used. However, the least-squares method has the limitation that it is only valid when the dependent variable (LMP in our case) follows the normal distribution. In this paper, we will illustrate how to use the least-squares method and the maximum likelihood method properly to fit any statistical distributions that the creep-rupture life may follow.

2 Model Description

In this section, we formalize the model structure for fitting the creep-rupture life distribution. The least-squares method is not necessarily the one for obtaining the materials constants in the equations. The fundamental assumptions that are used in this paper are as follows:

1 The basic Larson-Miller equation, as shown in Eq. (1), will be used.

2 The creep-rupture life is the random variable whose median or other location parameter is a function of temperature and stress. Temperature and stress are treated as control variables.

3 The average LMP can be represented by a polynomial function of log(S) as proposed by Conway (Eq. (2–34) in [6]).

Since t_r is probabilistic and T_c is deterministic, we know from Eq. (1) that LMP is also probabilistic. As a result, Eq. (2) should be interpreted as that the average of LMP can be represented by a second-order polynomial function of log(S). In the following, we will use the second-order polynomial function shown in Eq. (2) to illustrate the correct approaches for fitting statistical distributions of creep rupture life. The approaches to be discussed are also valid for a general polynomial function. To enable us to fit different statistical distributions to rupture life, we prefer the natural logarithm to common logarithm in the models. Equations (1) and (2) can be written in the following form when the natural logarithm is used instead:

$$\mathrm{LMP}' = (T_c + 273.2)(\ln t_r + C') \tag{5}$$

$$\mathrm{LMP}' = C_1' + C_2' \times \ln S + C_3' \times (\ln S)^2 \tag{6}$$

where LMP$'$, C', and C_i' for $i=1,2,3$ are the corresponding parameters when natural logarithm is used in the equations.

Equations (5) and (6) can be combined to generate the following equations:

$$\ln t_r = -C' + C_1' \frac{1}{T_c + 273.2} + C_2' \frac{\ln S}{T_c + 273.2} + C_3' \frac{(\ln S)^2}{T_c + 273.2} \tag{7}$$

$$\ln t_r = -C' + C_1' X_1 + C_2' X_2 + C_3' X_3 \tag{8}$$

where $X_i = (\ln S)^{i-1}/T_c + 273.2$ for $i=1,2$, and 3 and C', C_1', C_2', and C_3', are constants to be obtained from data analysis, to be discussed later. A general form of Eq. (8) when $f(S)$ is an nth-order polynomial function of log(S) is given by Conway [6].

The left side of Eq. (8) should be interpreted as the natural logarithm of the location parameter of the underlying distribution. More explanations will be provided in Section 3 using examples. Equation (2) uses the common logarithm, while Eq. (6) uses the natural logarithm. The relationship between the parameters in these two equations can be easily verified to be as follows:

$$\mathrm{LMP} = \mathrm{LMP}' \times \log e \tag{9}$$

$$C = C' \times \log e \tag{10}$$

$$C_1 = C_1' \times \log e \tag{11}$$

$$C_2 = C_2' \tag{12}$$

$$C_3 = C_3' / \log e \tag{13}$$

3 Approaches for Fitting Statistical Distributions of Creep-Rupture Life

The Least-Squares Method. The least-squares method assumes that LMP follows the normal distribution, or equivalently, t_r follows a lognormal distribution. It may be used to find the parameters in (8) from accelerated life testing data. Linear regression analysis is available in Microsoft Excel [7] and other spreadsheet software packages. If one uses the linear regression method, he/she must at least check the following to validate the model used:

1 Whether the fitted model shown in Eq. (8) can account for most of the variations in the observed creep-rupture life data; if the answer is no, one has to fit the data to an equation different from the one shown in Eq. (8). For example, one may expand Eq. (8) to include n covariables $X_1, X_2, \ldots, X_n$; i.e.,

$$\ln t_r = -C' + C_1' X_1 + C_2' X_2 + \ldots + C_n' X_n \tag{14}$$

where $n \geqslant 3$.

Whether the assumption that the natural logarithm of the creep-rupture life follows the normal distribution, or equivalently, whether the creep-rupture life follows the lognormal distribution; if the answer is no, then the least-squares method cannot be used to fit the mathematical relationship between LMP$'$ and $\ln S$. One has to try other methods which allow the creep-rupture life to follow other distributions.

Other tests may have to be conducted to validate the model obtained through the least-squares method; for example, whether some of the fitted parameters $C_i'(i=1,2,\ldots,n)$ are significantly different from 0. The readers may refer to Draper and Smith [8] for more details. In the following, we use the output from Microsoft Excel to illustrate the two foregoing critical tests.

We used Microsoft Excel 97 to analyze 109 accelerated creep-rupture life testing data points of low Si HP alloy produced by a manufacturer. Linear regression was used to fit the Eq. (7). The output from the software is shown in Fig. 1. The following is the fitted equation:

$$\ln t_{50} = -40.7534 + \frac{67844.79}{T_c + 273.2} - \frac{3376.28 \times \ln S}{T_c + 273.2} - \frac{1441.72 \times (\ln S)^2}{T_c + 273.2} \tag{15}$$

Because the natural logarithm of the creep-rupture life is assumed to follow the normal distribution, the fitted Eq. (15) relates the natural logarithm of the 50th percentile of the creep-rupture life to temperature and stress. From Fig. 1, we can see that the ''R Square'' of the model is equal to 0.9255, pretty close to 1. This means that 92.55 percent of the variations in the creep-rupture life data can be explained by the fitted model in Eq. (15). This pretty much confirms the validity of foregoing assumption 1. For utilization of other results in Fig. 1, readers are referred to [7] and [8].

In order to check the validity of the assumption that the creep-rupture life follows the lognormal distribution, we plotted a histogram of the residuals between the logarithms of the predicted life using Eq. (15) and the logarithms of the life data we had (Fig. 2). If the assumption is valid, the histogram should exhibit a symmetric, bell-shaped curve. However, Fig. 2 shows that the histogram is not symmetric; rather, it is skewed to the left. This shows that the second assumption is not satisfied. Thus, the fitted model shown in Eq. (15) is not valid.

It must be emphasized that the right statistical distribution is as important as the equation representing the relationship between the average of the logarithm of the creep-rupture life and the control variables, namely, stress and temperature. In reliability analysis or remaining life estimation of engineering devices, we are more interested in the remaining duration that the device can work reliably, say failure-free with a probability of 95 percent or

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.9620
R Square	0.9255
Adjusted R Squar	0.9233
Standard Error	0.3833
Observations	109

ANOVA

	df	SS	MS	F	Significance F
Regression	3	191.5314	63.8438	434.5051	5.0301E-59
Residual	105	15.4281	0.1469		
Total	108	206.9595			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	-40.7534	1.3973	-29.1648	3.66E-52	-43.5240	-37.9827	-43.5240	-37.9827
X1'	67844.7897	1928.4011	35.1819	6.31E-60	64021.1238	71668.4556	64021.1238	71668.4556
X2'	-3376.2791	624.2582	-5.4085	4.02E-07	-4614.0686	-2138.4896	-4614.0686	-2138.4896
X3'	-1441.7220	217.5114	-6.6283	1.5E-09	-1873.0072	-1010.4367	-1873.0072	-1010.4367

Fig. 1 Microsoft Excel output from linear regression analysis

higher. This means that we are more interested in the 5th or lower percentiles of the creep-rupture life, rather than the 50th percentile as shown in all the fitted equations from the least-squares method. If a wrong statistical distribution is used, large errors may result when one calculates the lower percentiles of the creep-rupture life. That is why the assumption of the lognormal distribution must be validated before the fitted model can be used in the assessment of the remaining life of an engineering device.

The Maximum Likelihood Method. The maximum likelihood method may be used to fit the parameters of any statistical distribution. Readers may refer to Freung and Walpole [9] for a discussion of this method. Suppose the creep-rupture life follows the Weibull distribution with the following:

$$\text{cumulative distribution function:}\quad F(t)=1-e^{-(t/\alpha)^{\beta}},\quad t>0 \tag{16}$$

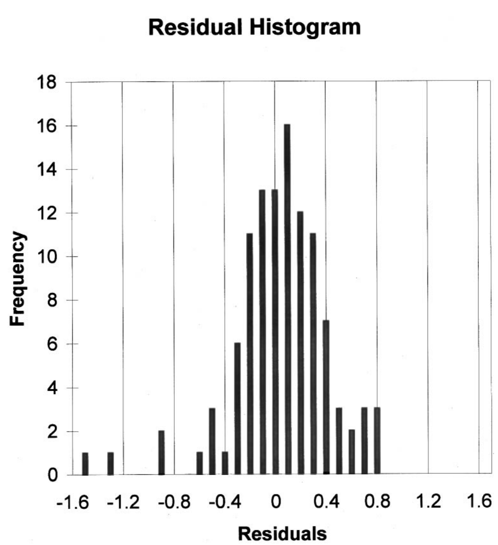

Fig. 2 Histogram of the residuals of the fitted model

$$\text{probability density function:}\quad f(t)=\frac{\beta t^{\beta-1}}{\alpha^{\beta}}e^{-(t/\alpha)^{\beta}},\quad t>0 \tag{17}$$

$$p\text{th percentile:}\quad t_p=\alpha[-\ln(1-p)]^{1/\beta} \tag{18}$$

where α and β are the location and shape parameters of the Weibull distribution, respectively. α is also called the characteristic life of t_r, and it is equal to the 63.2th percentile of the random variable t_r. The mean and variance of t_r are

$$E(t_r)=\alpha\Gamma\left(1+\frac{1}{\beta}\right) \tag{19}$$

$$\text{Var}(t_r)=\alpha^2\left(\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right) \tag{20}$$

where the Gamma function is defined as

$$\Gamma(z)=\int_0^{\infty}u^{z-1}e^{-u}du,\quad z>0 \tag{21}$$

The logarithm of the characteristic life α may be considered a function of temperature and stress in the form shown in Eq. (7). The shape parameter is assumed to be a constant. With n exact creep-rupture life data points, we can construct the following likelihood function if we feel that the rupture life may follow the Weibull distribution:

$$L(C',C_1',C_2',C_3',\beta)=\prod_{i=1}^{n}\left\{\frac{\beta t_{ri}^{\beta-1}}{(\alpha_i(C',C_1',C_2',C_3'))^{\beta}}\times\exp\left(-\left(\frac{t_{ri}}{\alpha_i(C',C_1',C_2',C_3')}\right)^{\beta}\right)\right\} \tag{22}$$

where $\alpha_i(C',C_1',C_2',C_3')=-C'+(C_1'/T_{ci}+273.2)+C_2'\ln S_i/(T_{ci}+273.2)+C_3'(\ln S_i)^2/(T_{ci}+273.2)$, T_{ci} and S_i are the temperature and stress, respectively, corresponding to the observed exact rupture life $t_{ri}(i=1,2,\ldots,n)$. The likelihood function in Eq. (22) is then maximized to find the parameters C', C_1', C_2', C_3', and β. A different likelihood function may be constructed if one suspects that the life may follow a different distribution. This new likelihood function may then be maximized to find another set of parameter estimates. The life distribution that gives the highest maximum likelihood function value provides the best representation of the observed creep rupture data. That distribution together with the parameter estimates should be used in future remaining life assessment of the materials in use.

If the lognormal distribution is assumed, the following functions should be used:

$$f(t)=\frac{1}{\sigma t\sqrt{2\pi}}e^{-1/2(\ln t-\mu/\sigma)^2},\quad t>0 \tag{23}$$

$$F(t)=\Phi\left(\frac{\ln t-\mu}{\sigma}\right),\quad t>0 \tag{24}$$

where $\Phi()$ is the cumulative distribution function of a standard normal random variable, σ is a constant, and μ is dependent on stress and temperature. The mean and variance of rupture life t_r are

$$E(t_r)=\text{MTTF}=e^{\mu+\sigma^2/2} \tag{25}$$

$$\text{Var}(t_r)=(e^{\sigma^2}-1)e^{\mu+\sigma^2} \tag{26}$$

where MTTF represents mean time to failure. The $100p$th lognormal percentile is

$$t_p=e^{\mu+z_p\sigma} \tag{27}$$

where z_p is the $100p$th percentile of a standard normal random variable. The 50th percentile $t_{50}=e^{\mu}$.

The LIFEREG procedure in SAS [10] may be used for fitting creep-rupture life distributions. The procedure uses the maximum likelihood method to determine the parameters of the considered statistical distribution for the rupture life. It can fit parametric models to failure data that may be right, left, or interval-censored. The models for the response variable, i.e., creep-rupture life in our study, consist of a linear effect of a few parameters with a few covariables, namely stress and temperature, together with a random disturbance term. The distribution of the random disturbance can be taken from a class of distributions that includes the extreme value, normal, and logistic distributions, and by using a log transformation, exponential, Weibull, lognormal, loglogistic, and gamma distributions. The advantage of this package is that there is a uniform structure for fitting all these different possible distributions in one run. The distribution with the largest maximum likelihood function value can then be selected to be the model of choice.

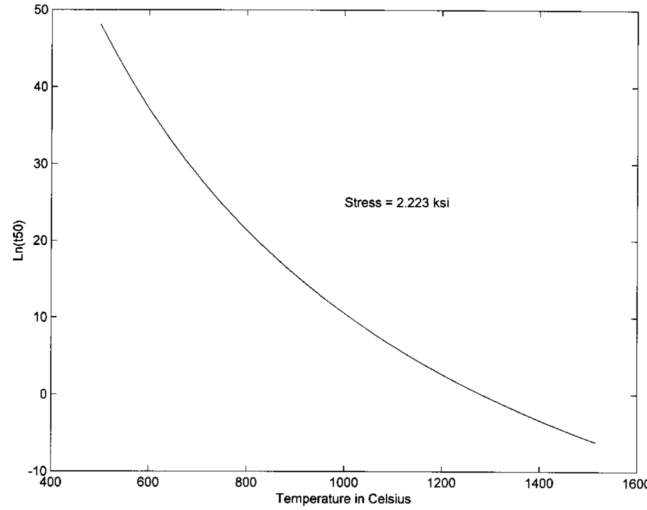

Fig. 4 The relationship between median life and temperature

With the accelerated rupture life testing data from the manufacturer of a heat-resistant alloy, we fit the failure data to the proposed model. Both Weibull and lognormal distributions were fit to the failure data. For the HP 40+Nb alloy, we found that the lognormal distribution provided a better fit. The fitted equations were

$$\ln t_{50}=-47.5747+\frac{78979.29}{T_c+273.2}-\frac{5089.16\times\ln S}{T_c+273.2}-\frac{1434.56\times(\ln S)^2}{T_c+273.2} \tag{28}$$

$$\text{LMP}'=(T_c+273.2)(\ln t_{50}+47.5747)/1000 \tag{29}$$

$$=78.9793-5.0892\times\ln S-1.4346\times(\ln S)^2 \tag{30}$$

Equivalently, when the common logarithm is used, we have

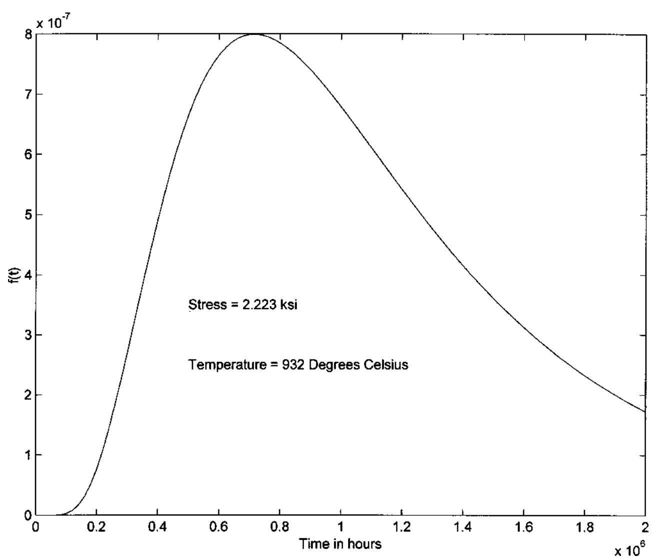

Fig. 3 Probability density function of the rupture life

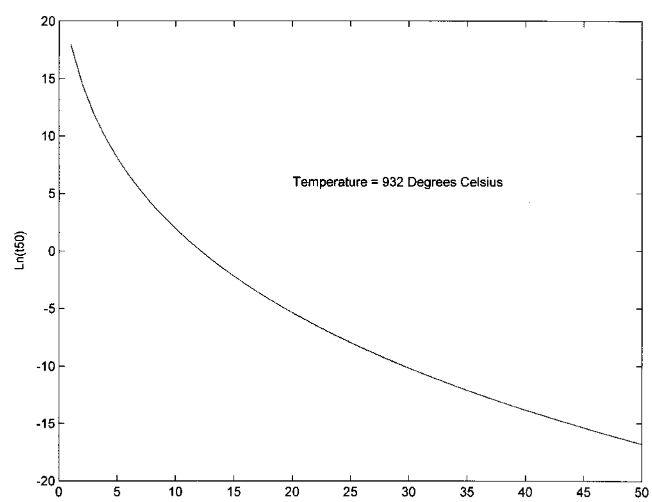

Fig. 5 The relationship between median life and stress

Table 1 Comparison of estimated lives by the manufacturer and the proposed model

Temperature in Celsius	Stress in ksi	Minimum Life given by Manufacturer	From the Proposed Model		
			2.5th Percentile	5th Percentile	10th Percentile
900	3.558	100,000	67,337	81,033	100,306
910	3.338	100,000	64,712	77,875	96,397
920	3.121	100,000	63,107	75,943	94,005
930	2.909	100,000	62,212	74,866	92,672
940	2.703	100,000	61,875	74,460	92,170
950	2.503	100,000	62,100	74,732	92,506
960	2.311	100,000	62,580	75,309	93,220
970	2.127	100,000	63,292	76,165	94,280
980	1.952	100,000	64,026	77,049	95,375
990	1.785	100,000	64,913	78,116	96,695
1000	1.628	100,000	65,494	78,816	97,562
1010	1.479	100,000	66,075	79,515	98,426
1020	1.34	100,000	66,120	79,569	98,493
1030	1.21	100,000	65,707	79,072	97,878
1040	1.09	100,000	64,458	77,569	96,018
1050	0.978	100,000	62,715	75,471	93,421
1060	0.875	100,000	60,124	72,353	89,562
1070	0.78	100,000	56,842	68,404	84,673
1080	0.694	100,000	52,595	63,293	78,347
1090	0.615	100,000	47,862	57,597	71,296
1100	0.543	100,000	42,683	51,364	63,581
1110	0.478	100,000	37,158	44,716	55,352
1120	0.42	100,000	31,455	37,853	46,856
1130	0.367	100,000	26,020	31,312	38,760

$$\log t_{50} = -20.6614 + \frac{34300.27}{T_c+273.2} - \frac{5089.16\times\log S}{T_c+273.2} - \frac{3303.21\times(\log S)^2}{T_c+273.2} \quad (31)$$

$$\mathrm{LMP} = (T_c+273.2)(\log t_{50}+20.6614)/1000 = 34.3003 - 5.0892\times\log S - 3.3032\times(\log S)^2 \quad (32)$$

The probability density function of the rupture life t_r under a temperature of 932°C and a stress of 2.223 ksi is shown in Fig. 3. The median life, t_{50}, as a function of temperature T_c given $S = 2.223$ ksi, is shown in Fig. 4. The median life t_{50} as a function of stress S given $T_c = 932$°C, is shown in Fig. 5. The percentiles of the rupture life obtained from the proposed model in this paper under the minimum rupture stress at given temperatures are compared with the 100,000-h rupture life provided by the manufacturers (Table 1). The calculated 2.5th rupture life percentiles are around 62,000 h, about 62 percent of the rupture life specified by the manufacturer when the temperature is 930°C and the stress is 2.909 ksi.

For the set of rupture life testing data for the 35C low Si alloy, we discovered that the rupture life followed the Weibull distribution. The obtained models are summarized in the following:

$$\beta = 8.1057 \quad (33)$$

$$\ln t_{63.2} = -44.9136 + \frac{73230.99}{T_c+273.2} - \frac{830.81\times\ln S}{T_c+273.2} - \frac{2395.7\times(\ln S)^2}{T_c+273.2} \quad (34)$$

$$\mathrm{LMP}' = (T_c+273.2)(\ln t_{63.2}+44.9136)/1000 = 73.2310 - 0.8308\ln S - 2.3957\times(\ln S)^2 \quad (35)$$

Equivalently, when the common logarithms were used, the fitted models were

$$\ln T_{63.2} = -19.5057 + \frac{31803.81}{T_c+273.2} - \frac{830.81\times\log S}{T_c+273.2} - \frac{5516.30\times(\log S)^2}{T_c+273.2} \quad (36)$$

$$\mathrm{LMP}' = (T_c+273.2)(\log t_{63.2}+19.5057)/1000 = 31.8038 - 0.8308\log S - 5.5163\times(\log S)^2 \quad (37)$$

4 Summary and Conclusion

This paper outlines the approaches for fitting creep-rupture life distributions. Most engineering literature in this area does not discuss the underlying assumptions used. It should be realized that sometimes these assumptions are not satisfied. With the points covered in this paper, the user should be able to find the right statistical distribution of the rupture life with the location parameter of the rupture life represented by a function of stress and temperature.

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References

[1] Boyer, H. E., 1988, *Atlas of Creep and Stress-Rupture Curves*, ASM International, Metal Park, OH 44073.

[2] Larson, F. R., and Miller, J., 1952, ''A Time-Temperature Relationship for Rupture and Creep Stress,'' Trans. ASME, **74**, p. 76.

[3] Jaske, C. E., and Simonen, F. A., 1991, ''Creep-Rupture Properties for Use in the Life Assessment of Fired Heater Tubes'', *Proc., First International Conference on Heat-Resistant Materials*, ASM International Materials Park, OH, pp. 485–493.

[4] American Petroleum Institute, 1988, ''Calculation of Heater-Tube Thickness in Petroleum Refineries,'' *API Recommended Practice 530*, 3rd Edition.

[5] Nelson, W., 1990, *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*, Wiley, New York, NY.
[6] Conway, J. B., 1969, *Stress-Rupture Parameters: Origin, Calculation and Use*, Gordon and Breach, New York, NY.
[7] Orvis, W. J., 1996, *Excel for Scientists and Engineers*, Sybex, San Francisco, CA.
[8] Draper, N. R., and Smith, H., 1981, *Applied Regression Analysis*, 2nd Edition, Wiley, New York, NY.
[9] Freung, J. E., and Walpole, R. E., 1980, *Mathematical Statistics*, 3rd Edition, Prentice-Hall, Englewood Cliffs, NJ.
[10] The SAS Institute, 1990, *SAS/STAT User's Guide*, Vol. 2, Version 6, Fourth Edition.