ELSEVIER

European Journal of Operational Research 123 (2000) 519–530

EUROPEAN
JOURNAL
OF OPERATIONAL
RESEARCH

www.elsevier.com/locate/dsw

Theory and Methodology

Replacement–repair policy for multi-state deteriorating products under warranty

Ming J. Zuo [a,*], Bin Liu [b], D.N.P. Murthy [c]

[a] *Department of Mechanical Engineering, University of Alberta, 4–9 Mechanical Engineering Building, Edmonton, Alberta, Canada T6G 2G8*
[b] *Chinese Academy of Sciences, Beijing, People's Republic of China*
[c] *The University of Queensland, Brisbane, Qld, Australia*

Received 1 March 1998; accepted 1 February 1999

Abstract

In this paper we study a warranty servicing policy for a class of multi-state deteriorating and repairable products. The manufacturer's decision to repair or replace a failed item depends on two parameters, the deterioration degree of the item and the length of the residual warranty period. We examine the optimal value of these two parameters to minimize the manufacturer's expected warranty servicing cost per item sold.

Keywords: Maintenance; Reliability; Multi-state; Warranty

---

1. Introduction

A warranty is a contract between the consumer and the manufacturer. The warranty normally specifies that the manufacturer agrees to replace or repair a failed item within a predetermined warranty period. The most commonly used warranty policies include the free replacement warranty (FRW), the pro-rata warranty, and the combined warranty (Murthy and Blischke, 1992). Under FRW, the manufacturer agrees to repair or provide replacements for failed items free of charge up to a time T from the time of initial purchase, where T is called the warranty period. If a product is under a pro-rata warranty, the replacements are provided under pro-rated cost to the consumer. Most of the warranty policies are combinations of simple policies. These policies are specific for different products and are referred to as combination policies.

With a given warranty policy, the manufacturer usually has the option of replacing a failed item with a new one or simply repairing it. Various research results have been reported on how a manufacturer should service a given warranty

---

[*] Corresponding author. Tel.: +1 780 492 4466; fax: +1 780 492 2200.
E-mail address: ming.zuo@ualberta.ca (M.J. Zuo).

PII: S 0 3 7 7 - 2 2 1 7 (9 9) 0 0 1 0 7 - 1

utilizing repair and/or replacement in order to minimize the total expected cost to the manufacturer during the warranty period. Nguyen and Murthy (1984) present models for evaluation of expected warranty cost and the variance of the warranty cost considering factors such as (1) general product lifetime distribution and time dependent repair cost, (2) good-as-new repair or replacement, (3) minimal repair, (4) mixture of replacement and minimal repair, and (5) imperfect repair. For the FRW with a fixed warranty period T, Nguyen and Murthy (1986) study another warranty servicing policy to minimize the expected warranty servicing cost. When an item fails and is returned to the manufacturer, the decisions that the manufacturer has to make are: (1) whether to replace a failed item by a new one or by a repaired item from the stock of repaired items; (2) whether to repair the failed item received or to discard it; and (3) the type of repair to be carried out. They propose an optimal servicing strategy for such a warranty policy. Nguyen and Murthy (1989) examine the combined FRW with fixed and renewed periods of T and W, respectively. They propose an optimal servicing strategy for the warranty policy. Rao (1995) develops algorithms for evaluating the total ownership cost and the cost to the manufacturer for products with the phase-type lifetime distributions under FRW. Bohoris and Yun (1995) provide equations for calculation of the expected value and variance of the manufacturer's total warranty costs under the combined warranty policy, minimal repair, and Weibull lifetime distributions. Chun and Tang (1995) determine the warranty price for the FRW policy assuming constant failure rate, constant repair cost throughout the warranty period, and the manufacturer's and the customer's risk aversion behavior for future repair costs. Murthy et al. (1995) propose a warranty policy which incorporates an incentive feature to reduce customer dissatisfaction to defective products. It involves the manufacturer offering compensation for consumers who experience early product failures as part of the warranty policy. They study the optimal design of such a policy considering product quality variations, the warranty policy, and the servicing strategy of the manufacturer.

All research results on warranty servicing strategy reviewed above assume that an item under warranty may experience only two possible states: working or failed. They do not address the deterioration of the item along time. The change in state of the item under warranty is modeled by Nguyen and Murthy (1984) using imperfect repair and by Nguyen and Murthy (1989) using replacement by another repaired item. In these cases, the repaired or replaced item may be considered in a different state because it has a failure rate function different from the one just before the failure. However, they do not model multi-state equipment or items due to natural deterioration.

Derman et al. (1978) study the optimal replacement problem of a component where there are n types of replacements available differing only in price and the failure rates of exponential life distributions. Assaf and Levikson (1982) and Assaf (1982) extend this model to arbitrary and phase-type life distributions. They all provide conditions under which there exist an optimal replacement policy with the property that category i replacement is selected if and only if the required operating time remaining falls within a specified time interval $(t_{i-1}, t_i]$, for $1 \leqslant i \leqslant n$. These models do not consider the natural deterioration of the component and are not suitable for consumer product warranties where normally the customer's item is either replaced with a new one or minimally repaired. The customer may not like his or her failed item being replaced with another used one.

In this paper we develop a warranty servicing model for a class of multi-state deteriorating products. Each item may experience N different working states (gradually deteriorating from state 1 to state N). It may fail from any working state. Therefore, there are N possible failure states. When an item fails during the warranty period, the manufacturer has the option of either repairing it using minimal repair or replacing it with a new one free of charge to the customers. The problem facing the manufacturer is to choose appropriate actions to minimize the total expected servicing cost during warranty. The manufacturer's decision on repair or replacement depends on two variables: the residual warranty period (from the present time to expiration of warranty) and the degree of deterioration of the failed item.

The remaining part of this paper is organized as follows. The descriptions and assumptions of the proposed model are given in Section 2. In Section 3, we consider a simple model with two working states that shows how we mathematically derive the warranty servicing cost and obtain the optimal replacement–repair parameters. In Section 4 we extend the simple model and discuss the warranty cost minimization problem in a general case. Conclusions are provided in Section 5.

2. Description of model

Consider an item or equipment that deteriorates with time. One can model the deterioration by either a continuous or a discrete variable. We follow the latter approach involving N discrete working states numbered from 1 through N. Working state 1 corresponds to a new item and the degree of deterioration increases with the working state so that working state j corresponds to greater deterioration than working state i if $j > i$. Once the item enters working state j, it can either fail or move to working state $j+1$. If it enters a failed state from working state j, then it can be made operational either through minimal repair or by replacement. In the former case, it is restored back to working state j and in the latter case it is brought back to working state 1. In working state N, when a failure occurs, the item is made operational by replacement so that the working state becomes 1 after replacement. Fig. 1 shows all the possible states of the item and the possible transitions among the states.

Models of deterioration based on the discrete approach has the advantage that they are simpler than the models based on the continuous approach and are easier to analyze. It can be viewed as an approximation to the real world where the deterioration is continuous as opposed to discrete. Lloyd and Lipow (1962) were one of early researchers advocating this approach to modeling deterioration. Since then, several researchers have used this approach to model unreliable items.

We model the changes in the states as follows. Once the item enters working state j it stays in that state for a random length of time which is given by an exponential distribution with transition parameter μ_j. This implies that the mean time to transition is $1/\mu_j$. The parameter μ_j decreases as j increases from 1 to N. The probability that it fails (and moves to failed state j) is given by p_j and the probability that it moves to working state $j+1$ is $1-p_j$.

The item is sold with a FRW policy with a warranty period T. This requires the manufacturer to either repair or provide replacements for failed items free of charge up to a time T from the time of the initial purchase. The warranty expires at time T after purchase.

We assume that the item is repairable and that the manufacturer has the option of either repairing a failed item or replacing it by a new one when it is returned under warranty. The repair or replacement time is assumed to be relatively short compared to the mean time between failures and hence can be treated as negligible. The optimal choice must be based on minimizing the expected cost. As

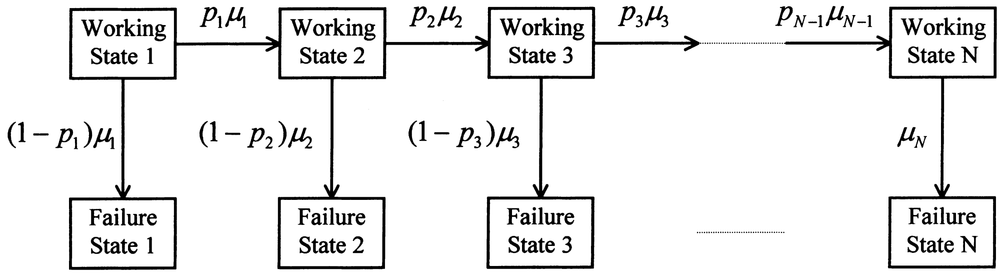

Fig. 1. State transition diagram of an item.

such we consider the following replacement–repair policy:

> If the failed item is in failure state i, $1 \leqslant i \leqslant N$, and the residual warranty time is t, $0 \leqslant t < T$, then it is replaced by a new one if and only if $K + 1 \leqslant i \leqslant N$ and $t \geqslant \alpha$; otherwise, it is minimally repaired, where $1 \leqslant K \leqslant N$ and $0 < \alpha \leqslant T$.

Note that the policy is characterized by two parameters – K and α. The manufacturer has to select the parameters K and α to minimize the expected cost of servicing the warranty. Under this policy, most failed items during the warranty period would normally be minimally repaired. However, if an item fails early in the warranty period and for some reason, the degree of deterioration is large, then it is more economical to replace it by a new one. As a result, the policy effectively avoids (a) unnecessary replacements when the failed item has only experienced minor deterioration and (b) excessive repairs when the failed item has already experienced heavy deterioration and the remaining warranty service time is still long.

Let $J(\alpha, K; T)$ denote the expected warranty servicing cost per item to the manufacturer under this warranty servicing policy. Then the problem is to determine the optimal K and α values which yield the minimum value for $J(\alpha, K; T)$.

We first consider the case $N = 2$ in which case $K \leqslant 1$ and derive an expression for $J(\alpha, K; T)$ (Section 3). We then extend the results to the case $N > 2$ in Section 4. Once an expression for $J(\alpha, K; T)$ is obtained, the optimal values are obtained using standard optimization techniques as will be indicated later in the paper.

We use the following notation:

N	number of working states and number of failure states,
T	length of the original warranty period,
K	a decision variable ($1 \leqslant K \leqslant N$),
α	a decision variable. ($0 < \alpha \leqslant T$),
$C_r^{(i)}$	cost of minimal repair given that the item is failed in state i ($i = 1, 2, \ldots, N$),
$C_m^{(i)}$	cost of replacing the failed item given that it failed in state i ($i = 1, 2, \ldots, N$),
μ_i	rate of transition from working state i ($i = 1, 2, \ldots, N$),
p_i	probability that the item enters working state $i + 1$ given that it has made a transition out of working state i ($i = 1, 2, \ldots, N - 1$),
$1 - p_i$	probability that the item enters failure state i given that it has made a transition out of working state i ($i = 1, 2, \ldots, N - 1$),
$J(\alpha, K; T)$	expected warranty servicing cost per item to the manufacturer.

In addition, we make the following assumptions:

$$\text{(A1)}\ (1 - p_1)\mu_1 < (1 - p_2)\mu_2 < \cdots < (1 - p_{N-1})\mu_{N-1} < \mu_N.$$

$$\text{(A2)}\ C_m^{(1)} \leqslant C_m^{(2)} \leqslant \cdots \leqslant C_m^{(N-1)} \leqslant C_m^{(N)}.$$

$$\text{(A3)}\ C_r^{(1)} < C_r^{(2)} < \cdots < C_r^{(N-1)} < C_r^{(N)}.$$

(A1) states that as the item deteriorates, it becomes more likely to make a transition to a failure state. Assumptions (A2) and (A3) imply that the item becomes more costly to replace and repair as the deterioration increases.

3. Special case ($N = 2$)

In this section, we confine our discussion to the case $N = 2$, i.e., the item has only two possible working states and two possible failure states. For this case, the replacement–repair policy has the simplest form with $K = 1$: the failed item is replaced by a new one if and only if it is in failure state 2 and the residual warranty period is not less than α; otherwise it is minimally repaired. The analysis of the method in this situation provides the basis for the approach used for the more general case ($N > 2$) in Section 4.

3.1. Preliminary results

Assume that at time $t = 0$, the item is new (or in working state 1). Let X and I represent the time to the first failure and the corresponding failure state, respectively. X and I are two dependent random variables with $X \in (0, \infty)$ and $I = \{1, 2\}$. In this subsection, our main task is to find their joint probability distribution which is useful in our later analysis.

Define

$$F_i(x) = Pr\{X \leqslant x \text{ and } I = i\}, \quad i = 1, 2, \tag{1}$$

$$f_i(x) = \frac{\mathrm{d}F_i(x)}{\mathrm{d}x}, \; i = 1, 2. \tag{2}$$

The expressions of $f_1(x)$ and $f_2(x)$ are as follows (the derivations are given in Appendix A):

$$f_1(x) = (1 - p_1)\mu_1 \mathrm{e}^{-\mu_1 t}, \tag{3}$$

$$f_2(x) = \begin{cases} p_1 \mu_1^2 t \mathrm{e}^{-\mu_1 t}, & \text{if } \mu_1 = \mu_2, \\ \dfrac{p_1 \mu_1 \mu_2}{\mu_2 - \mu_1}(\mathrm{e}^{-\mu_1 t} - \mathrm{e}^{-\mu_2 t}), & \text{if } \mu_1 \neq \mu_2. \end{cases} \tag{4}$$

Define

$$m_{f_1}(x) = \sum_{k=1}^{\infty} f_1^{(k)}(x), \tag{5}$$

$$m_{f_1+f_2}(x) = \sum_{k=1}^{\infty} [f_1 + f_2]^{(k)}(x), \tag{6}$$

where $f_1^{(k)}(x)$ and $[f_1 + f_2]^{(k)}(x)$ are the k-fold convolutions of $f_1(x)$ and $f_1(x) + f_2(x)$, respectively. We have (see Appendix A for the derivations)

$$m_{f_1}(x) = (1 - p_1)\mu_1 \mathrm{e}^{-p_1 \mu_1 x}, \tag{7}$$

$$m_{f_1+f_2}(x) = \frac{\mu_1 \mu_2}{\mu_2 + p_1 \mu_1} - p_1 \mu_1 \left(1 - \frac{\mu_1}{\mu_2 + p_1 \mu_1}\right) \mathrm{e}^{-(\mu_2 + p_1 \mu_1)x}. \tag{8}$$

3.2. Warranty servicing cost

In this subsection we will obtain the expected warranty servicing cost per item incurred by the manufacturer. It is the total of repair and replacement costs for the item over its warranty period (0, T].

Additional Notation:

$\mathscr{A}(t)$	The expected cost to the manufacturer for the remaining warranty period given that the item is currently in working state 1 and the remaining warranty period is t $(0 < t \leqslant T)$.
$\mathscr{B}(t)$	The expected cost to the manufacturer for the remaining warranty period given that the item has just entered failure state 2 and the remaining warranty period is t $(0 < t \leqslant T)$.

The expected total warranty servicing cost per item to the manufacturer is $\mathscr{A}(T)$. In the following, we derive an expression of $\mathscr{A}(t)$. Noting that the item has exponential sojourn time in each working state and its repair and replacement times are negligible, we have

$$\mathscr{A}(t) = \int_0^t [C_r^{(1)} + \mathscr{A}(t - x)] f_1(x)\,\mathrm{d}x + \int_0^t \mathscr{B}(t - x) f_2(x)\,\mathrm{d}x, \tag{9}$$

and

$$\mathscr{B}(t) = \begin{cases} C_m^{(2)} + \mathscr{A}(t), & \text{if } \; t \geqslant \alpha, \\ C_r^{(2)} + C_r^{(2)} \mu_2 t, & \text{if } \; t < \alpha. \end{cases} \tag{10}$$

We need to consider two different cases in order to obtain $\mathscr{A}(t)$.

Case 1: $t < \alpha$.

By substituting Eq. (10) into Eq. (9), we have

$$\mathscr{A}(t) = \int_0^t [C_r^{(1)} + \mathscr{A}(t - x)] f_1(x)\,\mathrm{d}x + \int_0^t C_r^{(2)} [1 + \mu_2 (t - x)] f_2(x)\,\mathrm{d}x, \tag{11}$$

which can be rewritten as

$$\mathscr{A}(t) = g(t) + \int_0^t \mathscr{A}(t - x) f_1(x)\,\mathrm{d}x, \tag{12}$$

where

$$g(t) = \left[C_r^{(1)}(1-p_1) - C_r^{(2)}p_1\frac{\mu_2}{\mu_1}\right](1-\mathrm{e}^{-\mu_1 t}) + C_r^{(2)}p_1\mu_2 t.$$

Eq. (12) can be written in the following form (Karlin and Taylor, 1975):

$$\mathscr{A}(t) = g(t) + \int_0^t g(t-x)m_{f_1}(x)\,\mathrm{d}x. \tag{13}$$

Eq. (13) can be easily verified as follows. Taking Laplace transformation on both sides of Eq. (12), we get

$$\tilde{\mathscr{A}}(s) = \frac{\tilde{g}(s)}{1-\tilde{f}_1(s)} = \tilde{g}(s) + \frac{\tilde{f}_1(s)\tilde{g}(s)}{1-\tilde{f}_1(s)}.$$

Noting that $(\tilde{f}_1(s))/(1-\tilde{f}_1(s))$ is the Laplace transformation of function $m_{f_1}(x)$, we can obtain Eq. (13).

Using $m_{f_1}(x)$ from Eq. (7) we have

$$\mathscr{A}(t) = C_r^{(2)}\mu_2 t + \frac{1}{p_1}\left[C_r^{(1)}(1-p_1) - C_r^{(2)}\frac{\mu_2}{\mu_1}\right](1-\mathrm{e}^{-p_1\mu_1 t}). \tag{14}$$

Case 2: $t \geqslant \alpha$.

From Eqs. (9) and (10), we have

$$\mathscr{A}(t) = \int_0^t [C_r^{(1)} + \mathscr{A}(t-x)]f_1(x)\,\mathrm{d}x + \int_0^{t-\alpha}[C_m^{(2)} + \mathscr{A}(t-x)]f_2(x)\,\mathrm{d}x + \int_{t-\alpha}^{t} C_r^{(2)}[1+\mu_2(t-x)]f_2(x)\,\mathrm{d}x.$$

Defining $\mathscr{G}(\hat{t}) = \mathscr{A}(\alpha+\hat{t})$ for all $\hat{t} \geqslant 0$, we have

$$\mathscr{G}(\hat{t}) = k(\hat{t}) + \int_0^{\hat{t}} \mathscr{G}(\hat{t}-x)(f_1(x)+f_2(x))\,\mathrm{d}x, \tag{15}$$

where

$$k(\hat{t}) = C_r^{(1)}\int_0^{\alpha+\hat{t}} f_1(x)\,\mathrm{d}x + \int_{\hat{t}}^{\alpha+\hat{t}} \mathscr{A}(\alpha+\hat{t}-x)f_1(x)\,\mathrm{d}x + C_m^{(2)}\int_0^{\hat{t}} f_2(x)\,\mathrm{d}x + C_r^{(2)}\int_{\hat{t}}^{\alpha+\hat{t}}[1+\mu_2(\alpha+\hat{t}-x)]f_2(x)\,\mathrm{d}x. \tag{16}$$

Because

$$\int_{\hat{t}}^{\alpha+\hat{t}} \mathscr{A}(\alpha+\hat{t}-x)f_1(x)\,\mathrm{d}x = \int_0^{\alpha} \mathscr{A}(x)f_1(\alpha+\hat{t}-x)\,\mathrm{d}x$$

and the expression of $\mathscr{A}(x)$ in interval $x \in (0,\ \alpha]$ is given by Eq. (14), $k(\hat{t})$ can be rewritten as

$$k(\hat{t}) = \begin{cases} U_1(\alpha)\hat{t}\mathrm{e}^{-\mu_1\hat{t}} + V_1(\alpha)\mathrm{e}^{-\mu_1\hat{t}} + W, & \text{if } \mu_1 = \mu_2, \\ U_2(\alpha)\mathrm{e}^{-\mu_1\hat{t}} + V_2(\alpha)\mathrm{e}^{-\mu_2\hat{t}} + W, & \text{if } \mu_1 \neq \mu_2, \end{cases} \tag{17}$$

where

$$\begin{aligned} W &= C_r^{(1)}(1-p_1) + C_m^{(2)}p_1, \\ U_1(\alpha) &= -C_m^{(2)}p_1\mu_1 + C_r^{(2)}p_1\mu_1^2\alpha, \\ V_1(\alpha) &= C_r^{(2)}[\mu_1\alpha - 1] - C_m^{(2)}p_1 + \frac{C_r^{(2)} - C_r^{(1)}(1-p_1)}{p_1}[\mathrm{e}^{-p_1\mu_1\alpha} - (1-p_1)], \\ U_2(\alpha) &= C_r^{(2)}\left[\left(1+\frac{p_1\mu_1}{\mu_2-\mu_1}\right)\mu_2\alpha - \frac{\mu_2}{\mu_1}\right] - C_m^{(2)}\frac{p_1\mu_2}{\mu_2-\mu_1} + \frac{C_r^{(2)}\mu_2 - C_r^{(1)}(1-p_1)\mu_1}{p_1\mu_1}[\mathrm{e}^{-p_1\mu_1\alpha} - (1-p_1)], \\ V_2(\alpha) &= \frac{p_1\mu_1}{\mu_2-\mu_1}\left[C_m^{(2)} - C_r^{(2)}\mu_2\alpha\right]. \end{aligned}$$

Although $k(\hat{t})$ has two different forms corresponding to $\mu_1 = \mu_2$ and $\mu_1 \neq \mu_2$, the following equation reveals their consistency:

$$\lim_{\mu_2\to\mu_1}\left(U_2(\alpha)\mathrm{e}^{-\mu_1\hat{t}} + V_2(\alpha)\mathrm{e}^{-\mu_2\hat{t}}\right) + W = U_1(\alpha)\hat{t}\mathrm{e}^{-\mu_1\hat{t}} + V_1(\alpha)\mathrm{e}^{-\mu_1\hat{t}} + W. \tag{18}$$

The proof of Eq. (18) can be completed by direct manipulation in which the following two results are used:

$$\lim_{\mu_2 \to \mu_1} \frac{\mu_1 \mathrm{e}^{-\mu_2 t} - \mu_2 \mathrm{e}^{-\mu_1 t}}{\mu_2 - \mu_1} = -\mu_1 t \mathrm{e}^{-\mu_1 t} - \mathrm{e}^{-\mu_1 t},$$
$$\lim_{\mu_2 \to \mu_1} \frac{\mathrm{e}^{-\mu_2 t} - \mathrm{e}^{-\mu_1 t}}{\mu_2 - \mu_1} = -t \mathrm{e}^{-\mu_1 t}.$$

The solution of Eq. (15) should be in the following form (Karlin and Taylor, 1975):

$$\mathscr{G}(\hat{t}) = k(\hat{t}) + \int_0^{\hat{t}} k(\hat{t} - x) m_{f_1 + f_2}(x) \, \mathrm{d}x. \tag{19}$$

Substituting $k(\cdot)$ from Eq. (17) and $m_{f_1+f_2}(\cdot)$ from Eq. (8) and simplifying, we have the following closed form expression for $\mathscr{G}(\hat{t})$:

$$\begin{aligned} \mathscr{G}(\hat{t}) = {} & C_r^{(1)} \frac{(1-p_1)\mu_1}{\mu_2 + p_1\mu_1} \left[\mu_2 \hat{t} + p_1\mu_1 \frac{1 - \mathrm{e}^{-(\mu_2 + p_1\mu_1)\hat{t}}}{\mu_2 + p_1\mu_1} \right. \\ & \left. + \frac{1 - \mathrm{e}^{-p_1\mu_1\alpha}}{p_1\mu_1} \left(\mu_2 + p_1\mu_1 \mathrm{e}^{-(\mu_2+p_1\mu_1)\hat{t}} \right) \right] \\ & + C_r^{(2)} \frac{\mu_2}{\mu_2 + p_1\mu_1} \left[(\mu_2 + p_1\mu_1)\alpha \right. \\ & \left. - \frac{1 - \mathrm{e}^{-p_1\mu_1\alpha}}{p_1\mu_1} \left(\mu_2 + p_1\mu_1 \mathrm{e}^{-(\mu_2+p_1\mu_1)\hat{t}} \right) \right] \\ & + C_m^{(2)} \frac{p_1\mu_1\mu_2}{\mu_2 + p_1\mu_1} \left[\hat{t} - \frac{1 - \mathrm{e}^{-(\mu_2+p_1\mu_1)\hat{t}}}{\mu_2 + p_1\mu_1} \right]. \end{aligned} \tag{20}$$

Remark. Although $k(\hat{t})$ (see Eq. (17)) has different forms for cases $\mu_1 = \mu_2$ and $\mu_1 \neq \mu_2$, we find that $\mathscr{G}(\hat{t})$ has the same form as shown in Eq. (20). Such a result is not surprising as one can see from Eq. (18).

For any given α value, the expected warranty cost function of the item can be written as

$$\mathscr{C}(\alpha) = \mathscr{A}(T) = \mathscr{G}(T - \alpha). \tag{21}$$

Note that $J(\alpha, K; T)$ is given by $\mathscr{C}(\alpha)$ or $\mathscr{A}(T)$. Since $K = 1$, the only variable for optimisation is α and this is obtained by minimising $J(\alpha, K; T)$. This can be obtained from the usual first order conditions (setting the derivative of $J(\alpha, K; T)$ with respect to α to zero). We now give conditions which characterize the optimal α.

3.3. Optimal policy

Differentiating $\mathscr{C}(\alpha)$ given by Eq. (21) with respect to α and simplifying yields

$$\begin{aligned} \frac{\mathrm{d}\mathscr{C}(\alpha)}{\mathrm{d}\alpha} = {} & \frac{\mu_2}{\mu_2 + p_1\mu_1} \left[1 - \mathrm{e}^{-(\mu_2 + p_1\mu_1)(T-\alpha)} \right] \\ & \times \left[-p_1\mu_1 (C_m^{(2)} - C_r^{(2)}) + (C_r^{(2)}\mu_2 \right. \\ & \left. - C_r^{(1)}(1-p_1)\mu_1)(1 - \mathrm{e}^{-p_1\mu_1\alpha}) \right]. \end{aligned} \tag{22}$$

We need to consider the following cases:

B1. When

$$C_m^{(2)} \geqslant C_r^{(2)} + \frac{C_r^{(2)}\mu_2 - C_r^{(1)}(1-p_1)\mu_1}{p_1\mu_1} (1 - \mathrm{e}^{-p_1\mu_1 T}),$$

we have $\mathrm{d}\mathscr{C}(\alpha)/\mathrm{d}\alpha \leqslant 0$ for all α ($\leqslant T$). This implies that $\mathscr{C}(\alpha)$ is a non-increasing function in α over interval $[0,\ T]$. Therefore, the optimal α is given by

$$\alpha^* = T. \tag{23}$$

B2. When

$$C_m^{(2)} < C_r^{(2)} + \frac{C_r^{(2)}\mu_2 - C_r^{(1)}(1-p_1)\mu_1}{p_1\mu_1} (1 - \mathrm{e}^{-p_1\mu_1 T}),$$

by solving $\mathrm{d}\mathscr{C}(\alpha)/\mathrm{d}\alpha = 0$, we have

$$\alpha^* = -\frac{1}{p_1\mu_1} \ln \left[1 - \frac{p_1\mu_1 (C_m^{(2)} - C_r^{(2)})}{C_r^{(2)}\mu_2 - C_r^{(1)}(1-p_1)\mu_1} \right] \tag{24}$$

and

$$\frac{\mathrm{d}\mathscr{C}(\alpha)}{\mathrm{d}\alpha} \begin{cases} < 0, & \text{if } \alpha < \alpha^*, \\ > 0, & \text{if } \alpha > \alpha^*. \end{cases} \tag{25}$$

We can summarize this as follows.

C1. If

$$C_m^{(2)} \geqslant C_r^{(2)} + \frac{C_r^{(2)}\mu_2 - C_r^{(1)}(1-p_1)\mu_1}{p_1\mu_1} (1 - \mathrm{e}^{-p_1\mu_1 T}),$$

the optimal replacement–repair policy is to repair all failures during the warranty period $(0,\ T]$ and do no replacement.

C2. If

$$C_m^{(2)} < C_r^{(2)} + \frac{C_r^{(2)}\mu_2 - C_r^{(1)}(1-p_1)\mu_1}{p_1\mu_1}(1 - \mathrm{e}^{-p_1\mu_1 T}),$$

the optimal policy is to replace the item with a new one if it is in failure state 2 and the remaining warranty period is longer than $\alpha^* \in (0,\ T]$, where α^* is given by Eq. (24), and to repair the item otherwise.

4. General case ($N > 2$)

We now consider the general case with $N \geqslant 3$. Our goal in this section is to develop an effective algorithm for finding the optimal parameters (K^* and α^*) for the warranty servicing policy.

4.1. Joint distribution of failure time and failure state

As before let X_i and I_i represent the time to the first failure and the corresponding failure state, respectively, under the assumption that the item is in working state i ($i = 1, 2, \ldots, N$) at time $t = 0$. Clearly, X_i and I_i are two dependent random variables where $X_i \in (0,\ \infty)$ and $I_i = i, i+1, \ldots, N$. Define

$$F_{ij}(x) = \Pr\{X_i \leqslant x \text{ and } I_i = j\},$$
$$j = i, i+1, \ldots, N; \quad 1 \leqslant i \le N$$
$$f_{ij}(x) = \frac{\mathrm{d}F_{ij}(x)}{\mathrm{d}x}, \ 1 \leqslant i \leqslant j \leqslant N.$$

The deterioration process of the item can be regarded as a Markov process with $2N$ possible states (see Fig. 1). There are N absorbing states which represent the N failure states of the item. The approach we use to derive $f_{ij}(t)$ is similar to that used in Section 3. Define

$$P_i(t) = \Pr\{\text{The item is in working state } i \text{ at time } t\}, \quad i = 1, 2, \ldots, N,$$
$$Q_i(t) = \Pr\{\text{The item is in failure state } i \text{ at time } t\}, \quad i = 1, 2, \ldots, N.$$

The Kolmogorov's equations for this Markov process are (Ross, 1983):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_N(t) \end{pmatrix} = \begin{pmatrix} -\mu_1 & & & \\ p_1\mu_1 & -\mu_2 & & \\ & \ddots & \ddots & \\ & & p_{N-1}\mu_{N-1} & -\mu_N \end{pmatrix}\begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_N(t) \end{pmatrix}, \tag{26}$$

$$\frac{\mathrm{d}Q_i(t)}{\mathrm{d}t} = (1-p_i)\mu_i P_i(t), \quad i = 1, \ldots, N-1, \tag{27}$$

$$\frac{\mathrm{d}Q_N(t)}{\mathrm{d}t} = \mu_N P_N(t). \tag{28}$$

If we assume that $P_k(0) = 1$, $P_i(0) = 0$ for all $i = 1, \ldots, k-1, k+1, \ldots, N$, and $Q_i(0) = 0$ for $i = 1, 2, \ldots, N$, then the $Q_j(t)$ obtained by solving Eqs. (26)–(28) should equal to $F_{kj}(t)$ for all $k \leqslant j \leqslant N$. In other words, solving Eqs. (26)–(28) under different initial conditions, we can obtain $F_{kj}(t)$ and $f_{kj}(t)$ for all $1 \leqslant k \leqslant j \leqslant N$.

4.2. Warranty servicing cost

The approach used in this section is similar to that used in Section 3.2. Let $\mathscr{A}_i(t)$ represent the expected cost to the manufacturer during the remaining warranty period given that the item is in working state i ($i = 1, 2, \ldots, N$) and the length of the remaining warranty period is t ($0 < t \leqslant T$). Then, the expected total warranty servicing cost per item to the manufacturer is given by $\mathscr{A}_1(T)$. In the following, we write the integral equations for $\mathscr{A}_i(t)$ and these are derived in a manner similar to that in Section 3.2. Two different cases need to be considered:

Case 1′: $t < \alpha$.

$$\mathscr{A}_i(t) = \sum_{j=i}^{N} \int_0^t [C_r^{(j)} + \mathscr{A}_j(x)] f_{ij}(t-x)\,\mathrm{d}x,$$
$$\text{for } i = 1, 2, \ldots, N. \tag{29}$$

Case 2′: $t \geqslant \alpha$.

$$\mathscr{A}_i(t) = \begin{cases} \psi_i(t) + \sum_{j=i}^{K} \int_{\alpha}^{t} [C_r^{(j)} + \mathscr{A}_j(x)] \\ f_{ij}(t-x)\,\mathrm{d}x + \sum_{j=K+1}^{N} \int_{\alpha}^{t} [C_m^{(j)} + \mathscr{A}_1(x)] \\ f_{ij}(t-x)\,\mathrm{d}x \quad \text{for } i = 1,2,\ldots,K, \\ \psi_i(t) + \sum_{j=i}^{N} \int_{\alpha}^{t} [C_m^{(j)} + \mathscr{A}_1(x)] \\ f_{ij}(t-x)\,\mathrm{d}x, \quad \text{for } \; i = K+1,\ldots,N, \end{cases} \tag{30}$$

where

$$\psi_i(t) = \sum_{j=i}^{N} \int_{0}^{\alpha} [C_r^{(j)} + \mathscr{A}_j(x)] f_{ij}(t-x)\,\mathrm{d}x,$$
$$\alpha \leqslant t \leqslant T \text{ and } i = 1,2,\ldots,N.$$

It is difficult to obtain closed form solutions of Eqs. (29) and (30). However, they can be easily evaluated numerically. Our goal is to calculate the warranty servicing cost $\mathscr{A}_1(T)$ $(T \geqslant \alpha)$. It is obvious that $\mathscr{A}_i(t)$ is a continuous function of t over interval (0, T). Thus, we propose the following algorithm for finding a numerical solution of $\mathscr{A}_i(t)$.

The expected warranty servicing cost $J(\alpha, K; T)$ is given by $\mathscr{A}_1(T)$. Note that this is a mixed optimization problem since α is a real variable and K is an integer. One way of obtaining the optimal values is as follows. For a fixed K, obtain the optimal α $(= a^*(K))$ by the usual method. Then K^*, the optimal K, is obtained by minimizing $J(\alpha^*(K), K; T)$. Note that this is an integer optimization problem. An exhaustive approach starting with $K = 1$ can be used to obtain K^*. Once this is obtained, the optimal α is given by $\alpha^* = \alpha^*(K^*)$.

In general, it is impossible to obtain an $J(\alpha, K; T)$ analytically. One needs to use a computational approach to obtain it. We propose the following algorithm to obtain it.

Algorithm. Let $\tau = T/h$ and $t_l = l\tau$, $l = 0, 1, \ldots, h$, where h determines the accuracy of the numerical solutions. The higher the value of h, the more accurate the numerical solution. Obviously, $t_0 = 0$ and $t_h = T$. Let $\alpha = \theta\tau$ for some fixed θ $(1 \leqslant \theta \leqslant h)$.

Step 1: For $l = 1, \ldots, \theta$, we can approximate Eq. (29) as follows:

$$\begin{aligned} \mathscr{A}_i(t_l) &= \sum_{j=i}^{N} \left[\sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} [C_r^{(j)} + \mathscr{A}_j(x)] f_{ij}(t_l - x)\,\mathrm{d}x \right] \\ &\approx \sum_{j=i}^{N} \left[\sum_{k=0}^{l-1} [C_r^{(j)} + \mathscr{A}_j(t_k)] f_{ij}(t_l - t_k)\tau \right], \\ & i = 1,2,\ldots,N, \end{aligned} \tag{31}$$

where $A_i(t_0) = A_i(0) \equiv 0$ for all $i = 1, 2, \ldots, N$.

Step 2: For $l = \theta + 1, \ldots, h$, we have can approximate Eq. (30) as follows:

$$\mathscr{A}_i(t_l) \approx \begin{cases} \psi_i(t_l) + \sum_{j=i}^{K} \left[\sum_{k=\theta}^{l-1} [C_r^{(j)} + \mathscr{A}_j(t_k)] \right. \\ \left. f_{ij}(t_l - t_k)\tau \right] + \sum_{j=K+1}^{N} \left[\sum_{k=\theta}^{l-1} [C_m^{(j)} + \mathscr{A}_1(t_k)] \right. \\ \left. f_{ij}(t_l - t_k)\tau \right] \quad \text{for } \; 1 \leqslant i \leqslant K, \\ \psi_i(t_l) + \sum_{j=i}^{N} \left[\sum_{k=\theta}^{l-1} [C_m^{(j)} + \mathscr{A}_1(t_k)] \right. \\ \left. f_{ij}(t_l - t_k)\tau \right], \quad \text{for } \; K+1 \leqslant i \leqslant N, \end{cases} \tag{32}$$

where

$$\psi_i(t_l) = \sum_{j=i}^{N} \left(\sum_{k=0}^{\theta-1} (C_r^{(j)} + \mathscr{A}_j(t_k)) f_{ij}(t_l - t_k)\tau \right),$$
$$l = \theta + 1, \ldots, h \text{ and } i = 1, 2, \ldots, N.$$

This is a very simple and crude algorithm. One can use more refined algorithms to solve the integral equations given in Eqs. (29) and (30) which give greater accuracy. Making use of the algorithm, one can easily calculate $\mathscr{A}_1(t_h)$, i.e., the warranty servicing cost $J(\alpha, K; T)$ or $\mathscr{A}_1(T)$, given any α and K values. The optimal values of these need to be obtained as indicated earlier.

4.3. Example

The algorithm described in the previous two subsections was used to solve the following numerical example: $N = 4$, $\mu_1 = 0.5$/year,

$\mu_2 = 2/\text{year}$, $\mu_3 = 3/\text{year}$, $\mu_4 = 3.5/\text{year}$, $p_1 = 0.9$, $p_2 = p_3 = 0.6$, $C_r^{(1)} = \$40$, $C_r^{(2)} = \$50$, $C_r^{(3)} = \$300$, $C_r^{(4)} = \$400$, $C_m^{(1)} = \$300$, $C_m^{(2)} = \$500$, $C_m^{(3)} = \$600$, $C_m^{(4)} = \$800$, and $T = 3$ years.

The joint probability density functions obtained with Eqs. (26)–(28) are:

$$f_{11}(t) = 0.05\mathrm{e}^{-0.5t}, \quad t \geqslant 0,$$
$$f_{12}(t) = 0.24 \times (\mathrm{e}^{-0.5t} - \mathrm{e}^{-2t}), \quad t \geqslant 0,$$
$$f_{13}(t) = 0.432 \times (0.4\mathrm{e}^{-0.5t} - \mathrm{e}^{-2t} + 0.6\mathrm{e}^{-3t}), \quad t \geqslant 0,$$
$$f_{14}(t) = 1.512 \times (0.2\mathrm{e}^{-0.5t} - \mathrm{e}^{-2t} + 1.8\mathrm{e}^{-3t} - \mathrm{e}^{-3.5t}), \quad t \geqslant 0,$$
$$f_{22}(t) = 0.8\mathrm{e}^{-2t}, \quad t \geqslant 0,$$
$$f_{23}(t) = 1.44 \times (\mathrm{e}^{-2t} - \mathrm{e}^{-3t}), \quad t \geqslant 0,$$
$$f_{24}(t) = 5.04 \times (\mathrm{e}^{-2t} - 3\mathrm{e}^{-3t} + 2\mathrm{e}^{-3.5t}), \quad t \geqslant 0,$$
$$f_{33}(t) = 1.2\mathrm{e}^{-3t}, \quad t \geqslant 0,$$
$$f_{34}(t) = 12.6 \times (\mathrm{e}^{-3t} - \mathrm{e}^{-3.5t}), \quad t \geqslant 0,$$
$$f_{44}(t) = 3.5\mathrm{e}^{-3.5t}, \quad t \geqslant 0.$$

The expected warranty servicing cost was calculated with the recursive algorithm given in Section 4.2. The effect of parameters K and α on the warranty cost are shown in Fig. 2. As can be seen, the optimal warranty servicing parameters were $K^* = 2$ and $\alpha^* = 0.5$ with a total warranty servicing cost of \$408. In words, to minimize the expected warranty servicing cost when the warranty period is 3 years, the manufacturer should minimally repair all failures except when the item is in failure state 3 and the remaining warranty period is longer than 0.5 years.

5. Conclusions

In this paper, we have developed an warranty servicing policy for a product with N working states and N failure states. The policy is characterized by two parameters. For the special case where $N = 2$, we have obtained analytical solutions for the optimal parameter values. When $N > 2$, we have proposed a computational procedure. The model can be used for minimizing the warranty servicing cost of multi-state deteriorating products using minimal repairs and replacement.

The model reported in this paper can be extended in several ways. Some of the many issues that need further study are listed below.

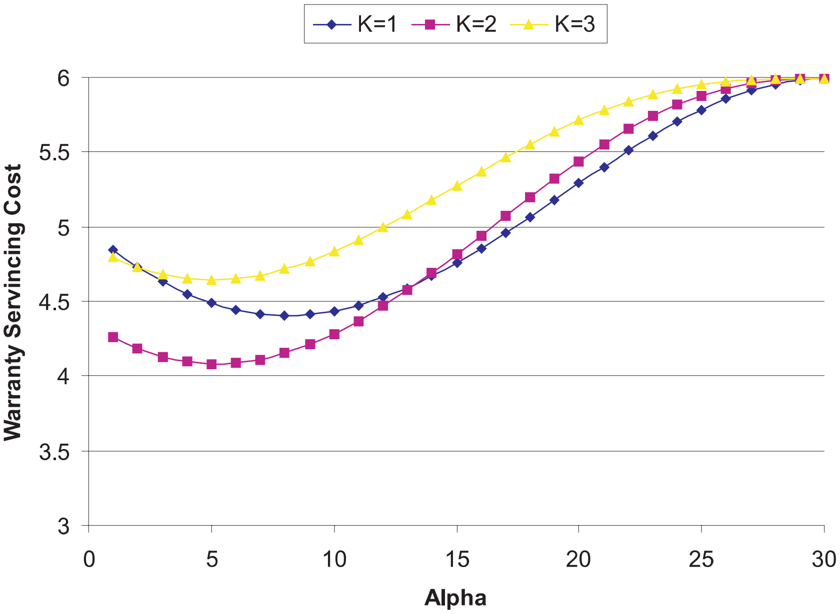

Fig. 2. Warranty servicing cost as a function of K and α.

(1) We have assumed that the sojourn times are exponential. One can relax this assumption and treat them as being more general distributions with parameters varying with the state.

(2) The FRW policy is one of many different warranty policies (Blischke and Murthy, 1994). One can carry out similar analysis to obtain the optimal repair–replacement strategies for other warranty policies. The equations to obtain $J(\alpha, K; T)$ would be more complex depending on the warranty policy.

(3) In our model the deterioration is characterized discretely and gradually through N states. A natural extension is to characterize the deterioration as a continuous variable. This will imply an infinite state space. One may also allow deteriorations from working state j to working state i for $i = j+1, j+2, \ldots, N$.

(4) We have assumed the (minimal) repair cost to be constant and independent of the failed state. One can relax this assumption so that (minimal) repair cost is a function of the failed state. Similarly, we can incorporate imperfect repair so that the number of attempts to rectify through repair is a random variable.

Acknowledgements

This research was partially supported by the University Grant Council of the Hong Kong Government. Part of this work was completed while Dr. Zuo was on sabbatical leave at the City University of Hong Kong.

Appendix A. Derivation of $f_1(x)$, $f_2(x)$, $m_{f_1}(x)$, and $m_{f_2}(x) (N = 2)$

The item is assumed to be new (in working state 1) at time 0. The deterioration process of the item can be regarded as a continuous-time Markov process. In order to derive the joint probability distribution of X and I, we treat the two possible failure states of the item as two absorbing states of this Markov process. The first failure time of the item is the time that the Markov process enters an absorbing state. The first failure state of the item is the absorbing state which this process enters. Define

$$P_i(t) = \Pr\{\text{The item is in working state } i \text{ at time } t\}, \quad i = 1, 2;$$

$$Q_i(t) = \Pr\{\text{The item is in failure state } i \text{ at time } t\}, \quad i = 1, 2.$$

Then, we have

$$F_i(t) = \Pr\{\text{The failure time of the item is less than or equal to } t \text{ and its failure state is } i\} = Q_i(t), \quad i = 1, 2. \tag{33}$$

Next, we write out the Kolmogorov's equations for this Markov process

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_1(t) \\ P_2(t) \\ Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} -\mu_1 & 0 & 0 & 0 \\ p_1\mu_1 & -\mu_2 & 0 & 0 \\ (1-p_1)\mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} P_1(t) \\ P_2(t) \\ Q_1(t) \\ Q_2(t) \end{pmatrix}, \tag{34}$$

with $P_1(0) = 1$ and $P_2(0) = Q_1(0) = Q_2(0) = 0$. Solving Eq. (34), we can obtain $Q_1(t)$ and $Q_2(t)$. Using Eq. (33), we obtain the expressions of $f_1(t)$ and $f_2(t)$ as shown in Eqs. (3) and (4).

Function $m_{f_1}(x)$ can be obtained directly from $f_1(x)$:

$$m_{f_1}(x) = \sum_{k=1}^{\infty} f_1^{(k)}(x) = \sum_{k=1}^{\infty} (1-p_1)^k \frac{\mu_1(\mu_1 x)^{k-1}}{(k-1)} \mathrm{e}^{-\mu_1 x} = (1-p_1)\mu_1 \mathrm{e}^{-p_1\mu_1 x}.$$

To derive $m_{f_1+f_2}(x)$, we first give the Laplace transform of functions $f_1(x)$ and $f_2(x)$ as shown in Eqs. (3) and (4):

$$\tilde{f}_1(s) = \frac{(1-p_1)\mu_1}{s+\mu_1},$$

$$\tilde{f}_2(s) = \frac{p_1\mu_1\mu_2}{(s+\mu_1)(s+\mu_2)}.$$

It should be noted that $\tilde{f}_2(s)$ has the above unified expression although $f_2(x)$ shown in Eq. (4) has different expressions for cases of $\mu_1 = \mu_2$ and $\mu_1 \neq \mu_2$.

The Laplace transform of $m_{f_1+f_2}(x)$ is given by

$$\tilde{m}_{f_1+f_2}(s) = \frac{\tilde{f}_1(s) + \tilde{f}_2(s)}{1 - (\tilde{f}_1(s) + \tilde{f}_2(s))} = \frac{(1-p_1)\mu_1 s + \mu_1\mu_2}{s(s+\mu_2+p_1\mu_1)}.$$

The reverse Laplace transform yields Eq. (8).

References

Assaf, D., 1982. Renewal decision when category life distributions are of phase-type. Mathematics of Operations Research 7 (4), 557–567.

Assaf, D., Levikson, B., 1982. On optimal replacement policies. Management Science 28 (11), 1304–1312.

Blischke, W.R., Murthy, D.N.P., 1994. Warranty Cost Analysis. Marcel Dekker, New York.

Bohoris, G.A., Yun, W.Y., 1995. Warranty costs for repairable products under hybrid warranty. IMA Journal of Mathematics Applied in Business and Industry 6, 13–24.

Chun, Y.H., Tang, K., 1995. Determining the optimal warranty price based on the producer's and customer's risk preferences. European Journal of Operational Research 85, 97–110.

Derman, C., Lieberman, G.J., Ross, S.M., 1978. A renewal decision problem. Management Science 24 (5), 554–561.

Karlin, S., Taylor, H.M., 1975. A First Course in Stochastic Processes. Academic Press, New York.

Lloyd, D.K., Lipow, M., 1962. Reliability: Management, Methods and Mathematics. Prentice-Hall, Englewood Cliffs, NJ.

Murthy, D.N.P., Blischke, W.R., 1992. Warranty management-III: A review of mathematical models. European Journal of Operational Research 62, 1–34.

Murthy, D.N.P., Djamaludin, I., Wilson, R.J., 1995. A consumer incentive warranty policy and servicing strategy for products with uncertain quality. Quality and Reliability Engineering International 11, 155–163.

Nguyen, D.G., Murthy, D.N.P., 1984. A general model for estimating warranty costs for repairable item. IIE Transactions 16, 379–386.

Nguyen, D.G., Murthy, D.N.P., 1986. An optimal policy for servicing warranty. Journal of Operational Research Society 11, 1081–1088.

Nguyen, D.G., Murthy, D.N.P., 1989. Optimal replace–repair strategy for servicing items sold under warranty. European Journal of Operational Research 39, 206–212.

Rao, B.M., 1995. Algorithms for the free replacement warranty with phase-type lifetime distributions. IIE Trans. 27, 348–357.

Ross, S.M., 1983. Stochastic Processes. Wiley, New York.