Sequential Imperfect Preventive Maintenance Models with Two Categories of Failure Modes

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Abstract: In this paper, we introduce the concept of two categories of failure modes: maintainable failure modes and nonmaintainable failure modes into the modeling of preventive maintenance (PM) activities. The hybrid model proposed by Lin et al. is used to model the effect of a PM activity on the failure rate function of maintainable failure modes. However, PM does not change the hazard rate function of nonmaintainable failure modes. PM is performed at a sequence of intervals. The objective is to determine the optimal PM schedules to minimize the mean cost rate. Numerical examples for Weibull hazard rates are given. © 2001 John Wiley & Sons, Inc. Naval Research Logistics 48: 172–183, 2001

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1. INTRODUCTION

Determination of a PM policy is an important issue in optimal maintenance planning. A PM policy specifies how PM activities should be scheduled. PM policies can be divided into two main categories: periodic and sequential. Periodic PM defines that a system is maintained at integer multiples of some fixed period and undergoes only minimal repair at failures between these PMs. Minimal repair only restores the function of the system when it is failed, but does not change the general health condition of the system. Sequential PM is the same as periodic PM except that the system is maintained at a sequence of intervals which may have unequal lengths. Periodic PM is more convenient to schedule, whereas sequential PM is more realistic when the system requires more frequent maintenance as it ages.

The general health condition of a piece of equipment has been measured with hazard rate function and effective age. The higher the hazard rate function value, the worse the equipment health

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condition. For equipment whose health condition deteriorates with its age, the term "effective age" has been used to measure its health condition. For deteriorating equipment, the higher the effective age, the worse the health condition of the equipment.

A PM model is used to measure the effects of a PM activity. Many publications have reported PM models. Traditional PM models assume that the system after PM is either "as good as new" or "as bad as old." The more realistic assumption is that the system after PM lies somewhere between as good as new and as bad as old. This kind of PM is called imperfect PM. There are two main alternatives for modeling an imperfect PM activity. The first one assumes that PM is equivalent to minimal repair with probability p and equivalent to replacement with probability $1-p$ [1–3, 10, 11, 16]. The second one models the PM's effects directly in terms of how the hazard rate function or the effective age of the equipment is changed by the PM [4–6, 8, 9, 13–15]. Using the concept of effective age, we may say that a certain PM has restored the health condition of a piece of equipment with a calendar age of 5 years to "as good as a 4-year old." Such a statement indicates that the PM has made the equipment younger and thus healthier.

Lie and Chun [6] and Nakagawa [12] introduce the concept of adjustment factors in hazard rate function and effective age to model the effects of PM. Nakagawa [15] proposes two PM models assuming that the required time for PM is negligible. Let's use t_i to represent the point of time when the ith PM is performed on the equipment $(i \geq 1)$. We will assume that a new piece of equipment is put in use at time $t_0 = 0$. These two PM models are described below.

1. **Hazard Rate PM Model:** The hazard rate function after the ith PM becomes $a_i h_{i-1}(x)$ for $x \in (0, t_{i+1} - t_i)$ when it was $h_{i-1}(x)$ for $x \in (0, t_i - t_{i-1})$, where $a_i > 1$ is the adjustment factor for the hazard rate function due to the ith PM. Nakagawa [15] also assumes that the hazard rate function is zero for a new piece of equipment. Based on these model assumptions, we can say that the hazard rate PM model represents the situation wherein the equipment's hazard rate function is an increasing function of time when there are no PM interventions, each PM resets the hazard rate function value to zero, and the rate of increase of the hazard rate function gets higher after each additional PM.
2. **Age Reduction PM Model:** The effective age after the ith PM reduces to $b_i E_i$ if the equipment's effective age was E_i just prior to this PM, where $b_i < 1$ is the improvement factor in the effective age of the equipment due to the ith PM. The effective age of equipment is the same as its actual age before the first PM is performed. In other words, at time t_1- (just prior to time t_1), the equipment's effective age and the actual age are both t_1. At time t_1+ (just after the first PM is completed), the effective age becomes $b_1 t_1$ while the actual age of the equipment remains the same. The equipment's health condition right after the first PM is represented by its hazard rate function value which is equal to the same value as that when the equipment's actual age was $b_1 t_1$. If we use $h_0(x)$ to represent the failure rate function prior to the first PM for $x \in (0, t_1)$, then $h_1(x) = h_0(b_1 t_1 + x)$ for $x \in (0, t_2 - t_1)$ represents the failure rate function of the equipment in the time interval of (t_1, t_2). The hazard rate function of the equipment is then a function of its effective age and each PM reduces the effective age of the equipment to a certain extent.

Based on the descriptions of these two models, we can say that the age reduction PM model has the advantage of determining the instantaneous failure rate function value right after a PM. In other words, we can say $h(t_1+) = h(b_1 t_1)$ if we use $h(t)$ to denote the hazard rate function of the equipment as a function its actual age. On the other hand, the hazard rate PM model has the advantage of allowing the rate of increase of the hazard function to be higher after each additional PM is performed on the equipment. Based on these observations, Lin, Zuo, and Yam [7] propose the Hybrid PM Model that combines the advantages of the Age Reduction PM Model and the Hazard Rate PM Model. In the Hybrid PM Model, it is assumed that effects of each PM are modeled by two aspects: one for its immediate effects after the PM is completed and the other for the lasting effects when the equipment is put to use again. This hybrid model captures both by how much the effective age is reduced the instant the PM is completed and how much faster the hazard rate function will increase after the equipment is put into use again. With this Hybrid PM Model, the failure rate function after the first PM may be written as

$$h(t_1 + x) = a_1 h(b_1 t_1 + x), \qquad \text{for } x > 0.$$

Using this Hybrid PM Model, we can still use the concept of effective age by saying that the effective age of the equipment becomes $b_1 t_1$ right after the first PM. However, the failure rate at this moment is not the same as the failure rate when the equipment's age is $b_1 t_1$. Instead, it is equal to $a_1 h(b_1 t_1)$. The adjustment factor a_1 that is used in the Hazard Rate PM Model is also used here to find the failure rate function value right after the PM is completed. Figure 1 is given for comparison of these three models before and after the first PM is performed assuming a linear hazard rate function when there is no PM intervention.

Two popular approaches used to determine the maintenance intervals for a sequential PM are the reliability-based method and the optimization method. For the former method, PM is performed whenever the system reliability (or the hazard rate of the system) reaches a predetermined level

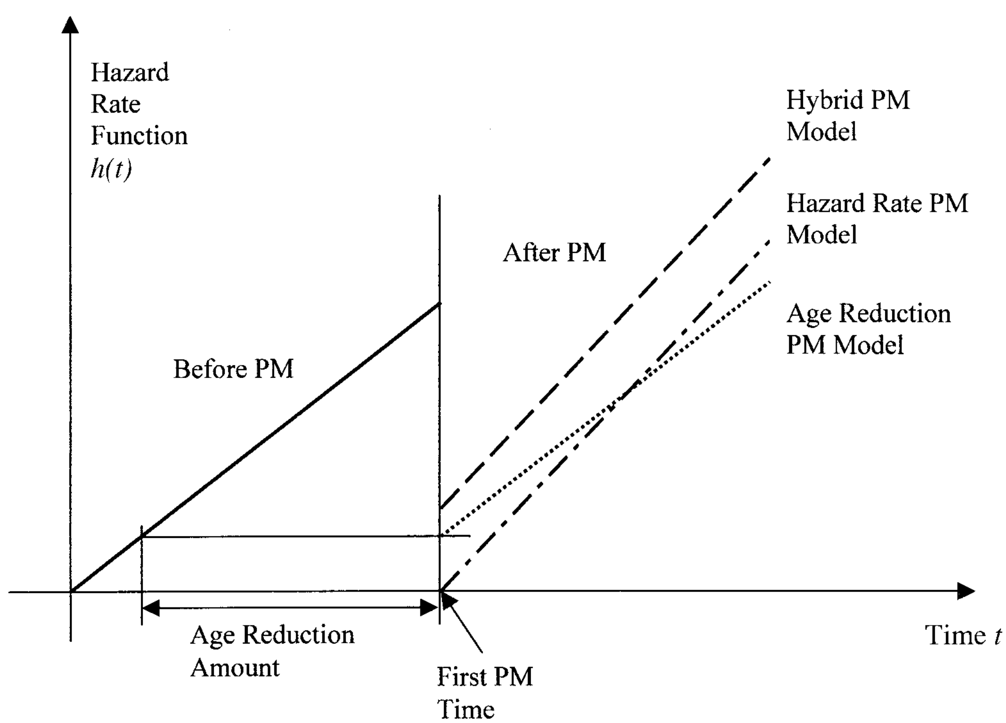

Figure 1. Comparison of the reported PM models.

[4–6, 8, 9]. The latter one selects the optimal intervals to minimize the mean cost rate or total life cycle cost [13–15].

It is reasonable to assume that system failure modes can be grouped into two categories: maintainable failure modes and nonmaintainable failure modes. PM can reduce the hazard rate of the maintainable failure modes but cannot change the hazard rate of the nonmaintainable failure modes. For example, PM of a machine may include oiling, tightening screws, and cleaning, etc. But this kind of PM activities can only improve the failure modes which are affected by such working environment of the machine. Those failure modes which are affected only by the inherent design of the machine, however, cannot be improved by the PM activities. In this paper, we introduce the concept of these two categories of failure modes to model PM activities. We also use the Hybrid PM Model to model the effects of PM on the maintainable failure modes. We develop optimal PM intervals through two alternatives: (1) leaving the PM intervals as decision variables and (2) allowing the PM intervals to be determined by the hazard rate or reliability limit. The objective functions in both cases are to minimize the mean cost rate. Numerical examples for the case of Weibull hazard rates will be given for illustration purposes.

Notation:

$h_a(t)$ = hazard rate of the nonmaintainable failure modes of the system,
$h_b(t)$ = hazard rate of the maintainable failure modes of the system,
$H_a(t)$ = cumulative hazard rate of the nonmaintainable failure modes of the system,
$H_b(t)$ = cumulative hazard rate of the maintainable failure modes of the system,
$h_k(t)$ = hazard rate of the system between the $(k-1)$th PM and the kth PM,
$H_k(t)$ = cumulative hazard rate of the system between the $(k-1)$th and the kth PMs,
λ = predetermined acceptable level of hazard rate,
x_k = scheduled PM intervals, $k = 1, 2, \ldots, N$,
t_k = $x_1 + x_2 + \cdots + x_k$, $k = 1, 2, \ldots, N$,
y_k = effective age of the system just before the kth PM, $k = 1, 2, \ldots, N$,
N = scheduled number of PM intervals (the Nth PM is actually a replacement),
a_k = adjustment factor in hazard rate of the maintainable failure modes after the kth PM, $1 = a_0 \le a_1 \le a_2 \le \cdots \le a_{N-1}$,
A_k = $\prod_{i=0}^{k-1} a_i$, $k = 1, 2, \ldots, N$
b_k = adjustment factor in effective age due to the kth PM, $0 = b_0 \le b_1 \le b_2 \le \cdots \le b_{N-1} < 1$,
c_m = cost of minimal repair,
c_p = cost of PM,
c_r = cost of replacement,
C = mean cost rate of the system,
d_k = $[\frac{(1-b_k)^\alpha}{\beta_1 + A_k\beta_2 - (\beta_1 + A_{k+1}\beta_2)b_k^\alpha}]^{1/(\alpha-1)}$.

Assumptions:

1. The planning horizon is infinite.
2. The hazard rate functions of the nonmaintainable and maintainable failure modes of the system, $h_a(t)$ and $h_b(t)$, are continuous and strictly increasing if there are no PM interventions.
3. The times for PM, minimal repair and replacement are negligible.

4. PM is performed at $t_1, t_2, \ldots, t_{N-1}$, and the system is replaced at t_N.
5. The system is restored to as good as new state at replacement.
6. Maintainable failure modes and non-maintainable failure modes are independent and they are competing to cause system failure.
7. The Hybrid PM Model proposed by Lin, Zuo, and Yam [7] will be used to model the effects of PM on maintainable failure modes.

2. MODEL FORMULATIONS AND OPTIMAL SOLUTIONS

We consider the situation where a system is preventively maintained at $t_1, t_2, \ldots, t_{N-1}$ and replaced at t_N. Minimal repair is performed at failures between PMs. Replacement of the system restores the system to the as good as new state. The system has the hazard rate due to maintainable failure rate $A_k h_b(t)$ between the $(k-1)$th and the kth PMs, i.e., in time interval (t_{k-1}, t_k). The hazard rate of the nonmaintainable failure rate in this time interval is $h_a(t)$. Thus, the hazard rate of the system in time interval (t_{k-1}, t_k) is $h_k(t) = h_a(t) + A_k h_b(t)$. Based on the model descriptions, we have

$$y_k = x_k + b_{k-1}x_{k-1} + \cdots + b_{k-1}b_{k-2}\cdots b_2 b_1 x_1,$$

$$y_k = x_k + b_{k-1}y_{k-1},$$

$$x_k = y_k - b_{k-1}y_{k-1}.$$

Hence the mean cost rate is

$$C = \frac{c_r + c_p(N-1) + c_m \sum_{k=1}^{N}[H_k(y_k) - H_k(b_{k-1}y_{k-1})]}{[\sum_{k=1}^{N-1}(1-b_k)y_k + y_N]}. \tag{1}$$

There are two main alternatives to determine the PM intervals in the literature. One is to select optimal PM intervals to minimize the mean cost rate, i.e., to leave PM intervals as decision variables in the optimization problem [4–6, 8, 9]. The other is to determine PM intervals by the hazard rate or reliability limit [13–15]. They will be used respectively in the following model formulations.

2.1. Model 1

In this model, the PM times $t_1, t_2, \ldots, t_{N-1}$ and the replacement time t_N are all treated as independent decision variables. Alternatively, the decision variables are N and y_k $(k = 1, 2, \ldots, N)$. That is, the objective in this model is to determine the optimal values of N and y_k $(k = 1, 2, \ldots, N)$ to minimize the mean cost rate expressed in Eq. (1).

Setting the differentiation of Eq. (1) with respect to y_k to zero leads to

$$h_k(y_k) - b_k h_{k+1}(b_k y_k) = (1-b_k)[h_a(y_N) + A_N h_b(y_N)], \qquad k = 1, 2, \ldots, N-1, \tag{2}$$

and

$$c_m[h_a(y_N) + A_N h_b(y_N)] = C. \tag{3}$$

THEOREM 1: For a fixed y_N $(0 < y_N < \infty)$, the solution to Eq. (2) with respect to y_k $(y_k > 0)$ exists if $1 - a_k b_k > 0, k = 1, 2, \ldots, N-1$. Further, the solution is unique if $h_a(t)$ and $h_b(t)$ are differentiable and $h'_a(t)$ and $h'_b(t)$ are strictly increasing.

PROOF: If $1 - a_k b_k > 0$, we can show that the lhs of Eq. (2) is less than the rhs when $y_k = 0$:

$$h_k(0) - b_k h_{k+1}(0) = (1 - b_k)h_a(0) + A_k(1 - a_k b_k)h_b(0) < (1 - b_k)[h_a(y_N) + A_N h_b(y_N)].$$

With $1 - a_k b_k > 0$, the lhs of Eq. (2) is greater than the rhs when $y_k \to +\infty$:

$$h_k(y_k) - b_k h_{k+1}(b_k y_k) \geq (1 - b_k)h_a(y_k) + A_k(1 - a_k b_k)h_b(y_k) \to +\infty \qquad \text{as } y_k \to +\infty.$$

It should be noted that the rhs is a constant for a fixed y_N. Therefore, there exists some y_k $(y_k > 0)$ which satisfies Eq. (2). Moreover, when $h'_a(t)$ and $h'_b(t)$ are strictly increasing, the derivative of the lhs of Eq. (2) with respect to y_k is

$$h'_k(y_k) - b_k^2 h'_{k+1}(b_k y_k) > (1 - b_k^2)h'_a(y_k) + A_k(1 - a_k b_k)h'_b(y_k) > 0.$$

That is, the lhs of Eq. (2) is a strictly increasing function of y_k. Therefore, the solution to Eq. (2) with respect to y_k is unique.

Note that the condition $1 - a_k b_k > 0$ means that the hazard rate adjustment factor a_k (> 1) should be smaller than the reciprocal of the age reduction factor b_k. This is a reasonable condition since PM tends to improve the system's condition. The extent of system improvement by PM is determined by the values of a_k and b_k: The smaller the values of a_k and b_k, the better the improvement. If the condition $1 - a_k b_k > 0$ does not hold, the solution to Eq. (2) may not exist. Then the optimal y_k may be attained at boundary points or may not exist. In the following, we assume that the condition $1 - a_k b_k > 0$ holds.

Substituting each solution to Eq. (2), y_k $(k = 1, 2, \ldots, N-1)$, into Eq. (3), we have

$$h_N(y_N)\left[\sum_{k=1}^{N-1}(1 - b_k)y_k + y_N\right] - \sum_{k=1}^{N}[H_k(y_k) - H_k(b_{k-1}y_{k-1})] = \frac{c_r + c_p(N-1)}{c_m}, \tag{4}$$

where each y_k $(k = 1, 2, \ldots, N-1)$ is some function of y_N. Then, the lhs of Eq. (4) is a function only of y_N. □

THEOREM 2: If $1 - a_k b_k > 0, k = 1, 2, \ldots, N-1, h_a(t)$ and $h_b(t)$ are differentiable, $h_a(0) = 0, h_b(0) = 0$, and $h'_a(t)$ and $h'_b(t)$ are strictly increasing, the solution to Eq. (4) with respect to y_N $(y_N > 0)$ exists and is unique.

PROOF: Since

$$h_k(y_k) - b_k h_{k+1}(b_k y_k) \geq (1 - b_k)h_a(y_k) + A_k(1 - a_k b_k)h_b(y_k) \geq 0, \qquad k = 1, 2, \ldots, N-1,$$

the solution to Eq. (2) with respect to y_k is zero when $y_N = 0$. Thus, if $y_N = 0$, the lhs of Eq. (4) equals zero and is smaller than the rhs of Eq. (4). On the other hand, the derivative of the lhs of Eq. (4) with respect to y_N is found to be

$$[h'_a(y_N) + A_N h'_b(y_N)]\left[\sum_{k=1}^{N-1}(1 - b_k)y_k + y_N\right] > 0$$

if $y_N > 0$. Hence, the lhs of Eq. (4) is a strictly increasing function of y_N $(y_N \geq 0)$ and equals zero at $y_N = 0$. Therefore, the solution to Eq. (4) with respect to y_N $(y_N > 0)$ exists and is unique.

Based on the above results, we obtain the following algorithm for finding the optimal PM schedule: □

Step 1. Solve Eq. (2) with respect to y_k $(k = 1, 2, \ldots, N-1)$. The solutions are functions of y_N.

Step 2. Substitute the solutions in Step 1 into Eq. (4) and solve it with respect to y_N.

Step 3. Choose N to minimize $[h_a(y_N) + A_N h_b(y_N)]$, where y_N is the solution obtained in Step 2.

Step 4. Compute $y_k, k = 1, 2, \ldots, N$ from expressions obtained in Steps 1 and 2 for the value of N obtained in Step 3.

Step 5. Compute x_k from $x_k = y_k - b_{k-1} y_{k-1}, k = 1, 2, \ldots, N$.

When the condition $h_a(0) = 0$ and $h_b(0) = 0$ in Theorem 2 does not hold, the solution to Eq. (4) with respect to y_N $(y_N > 0)$ may or may not exist. But the above algorithm still works if the solution to Eq. (4) with respect to y_N $(y_N > 0)$ exists.

It should be noted that Model 1 reduces to a pure hazard rate reduction model when $b_k = 0, k = 1, 2, \ldots, N$, and a pure age reduction model when $a_k = 1, k = 1, 2, \ldots, N$.

2.2. Model 2

In this model, PM is performed whenever the hazard rate of the system reaches the predetermined level λ and the decision variable is N. This implies that the hazard rate at time t_i $(i = 1, 2, \ldots, N)$ must equal λ. That is,

$$h_k(y_k) = h_a(y_k) + A_k h_b(y_k) = \lambda, \qquad k = 1, 2, \ldots, N. \tag{5}$$

Solving Eq. (5) with respect to y_k $(k = 1, 2, \ldots, N)$, we get the solutions as expressions of λ. Substitute these expressions of λ. Substitute these expressions into Eq. (1). Then, C becomes a function of λ. Differentiating C with respect to λ and setting it to zero leads to

$$\frac{\sum_{k=1}^{N-1} \frac{h_k(y_k) - b_k h_{k+1}(b_k y_k)}{h'_k(y_k)} + \frac{h_N(y_N)}{h'_N(y_N)}}{\sum_{k=1}^{N-1} \frac{1-b_k}{h'_k(y_k)} + \frac{1}{h'_N(y_N)}} = \frac{C}{c_m}, \tag{6}$$

where y_k $(k = 1, 2, \ldots, N)$ is a function of λ. Solving Eq. (6) with respect to λ, we obtain λ as a function of N. Then, we can find N which minimizes

$$\frac{\sum_{k=1}^{N-1} \frac{h_k(y_k) - b_k h_{k+1}(b_k y_k)}{h'_k(y_k)} + \frac{h_N(y_N)}{h'_N(y_N)}}{\sum_{k=1}^{N-1} \frac{1-b_k}{h'_k(y_k)} + \frac{1}{h'_N(y_N)}}, \tag{7}$$

where y_k $(k = 1, 2, \ldots, N)$ is a function of λ and thus a function of N.

Based on the above discussions, we derive the algorithm for obtaining the optimal PM schedule as follows.

Step 1. Solve Eq. (5) with respect to y_k $(k = 1, 2, \ldots, N-1)$. The solutions are functions of λ.

Step 2. Substitute the solutions in Step 1 into Eq. (6) and solve it with respect to λ.

Step 3. Choose N to minimize the function in expression (7) where y_k $(k = 1, 2, \ldots, N)$ is the expression obtained in Steps 1 and 2.

Step 4. Compute $y_k, k = 1, 2, \ldots, N$ from expressions obtained in Steps 1 and 2 for the value of N obtained in Step 3.

Step 5. Compute x_k from $x_k = y_k - b_{k-1}y_{k-1}, k = 1, 2, \ldots, N$.

It can be shown that Eqs. (5) and (6) are equivalent to Eqs. (2) and (3), respectively, when $b_k = 0, k = 1, 2, \ldots, N$. This means that the optimal PM schedules in Model 1 and Model 2 are the same for a pure hazard rate reduction policy.

3. NUMERICAL EXAMPLES

We consider the case when the hazard rate functions $h_a(t)$ and $h_b(t)$ of the system are Weibull and have the same shape parameter α, i.e.,

$$h_a(t) = \beta_1 t^{\alpha-1}, \qquad h_b(t) = \beta_2 t^{\alpha-1}, \qquad \alpha > 1, \quad \beta_1 > 0, \quad \beta_2 > 0.$$

For Model 1, Eq. (2) becomes

$$\beta_1 y_k^{\alpha-1} + A_k\beta_2 y_k^{\alpha-1} - b_k[\beta_1(b_k y_k)^{\alpha-1} + A_{k+1}\beta_2(b_k y_k)^{\alpha-1}] = (1-b_k)(\beta_1 y_N^{\alpha-1} + A_N\beta_2 y_N^{\alpha-1}),$$

for $k = 1, 2, \ldots, N-1$. Solving this equation we get

$$y_k = \left[\frac{(1-b_k)(\beta_1 + A_N\beta_2)}{\beta_1 + A_k\beta_2 - (\beta_1 + A_{k+1}\beta_2)b_k^{\alpha}}\right]^{1/(\alpha-1)} y_N, \qquad k = 1, 2, \ldots, N-1. \tag{8}$$

Substituting equation (8) into equation (3) leads to

$$y_N = \frac{[c_r + c_p(N-1)]^{1/\alpha}}{\{c_m(1-1/\alpha)[\beta_1 + A_N\beta_2 + (\beta_1 + A_N\beta_2)^{\alpha/(\alpha-1)}\sum_{k=1}^{N-1} d_k]\}^{1/\alpha}}. \tag{9}$$

Then, $h_a(y_N) + A_N h_b(y_N)$ becomes

$$\frac{(\beta_1 + A_N\beta_2)[c_r + c_p(N-1)]^{(\alpha-1)/\alpha}}{\{c_m(1-1/\alpha)[\beta_1 + A_N\beta_2 + (\beta_1 + A_N\beta_2)^{\alpha/(\alpha-1)}\sum_{k=1}^{N-1} d_k]\}^{(\alpha-1)/\alpha}}.$$

Minimization of $h_a(y_N) + A_N h_b(y_N)$ is equivalent to minimization of function

$$B(N) = \frac{c_r + c_p(N-1)}{(\beta_1 + A_N\beta_2)^{-1/(\alpha-1)} + \sum_{k=1}^{N-1} d_k}, \qquad N = 1, 2, \ldots.$$

The optimal value of N can be found by solving inequalities $B(N+1) \geq B(N)$ and $B(N) < B(N-1)$. These two inequalities imply that

$$D(N) \geq \frac{c_r}{c_p} \qquad \text{and} \qquad D(N-1) < \frac{c_r}{c_p}, \tag{10}$$

where

$$D(N) = \frac{(\beta_1 + A_N\beta_2)^{-1/(\alpha-1)} + \sum_{k=1}^{N-1} d_k}{(\beta_1 + A_{N+1}\beta_2)^{-1/(\alpha-1)} - (\beta_1 + A_N\beta_2)^{-1/(\alpha-1)} + d_N} - (N-1).$$

Thus, the optimal value N^* must satisfy inequalities (10). The optimal PM intervals are computed from $x_k = y_k - b_{k-1}y_{k-1}$ $(k = 1, 2, \ldots, N)$, where y_k $(k = 1, 2, \ldots, N)$ are given in Eqs. (8) and (9).

To compute the optimal PM schedule, we need to know the cost parameters c_m, c_p, and c_r, the Weibull parameters α, β_1, and β_2, and the adjustment factors a_k and b_k. For the cost parameters, actually we only need to know the cost ratios c_r/c_p and c_m/c_p. One has to obtain these parameters from data analysis using the failure and maintenance database. For example, a goodness-of-fit test needs to be conducted to see if the Weibull distribution can be used and if yes we need to estimate the parameters of the Weibull distribution. As far as the adjustment parameters are concerned, the users need to observe the failure rate function before and after each PM. The failure rate function values after a fixed amount of time has elapsed from every PM can be compared to fit the trend of the failure rate as a function of the number of PM activities. In our numerical example in this section, however, we will arbitrarily select these parameter values to illustrate the model and results presented earlier in this paper.

Table 1 shows the optimal PM schedules, $N^*, x_1, x_2, \ldots, x_{N^*}$, in Model 1 with and without nonmaintainable failure modes considered, respectively, for $c_r/c_p = 2, 5, 10, 20, 50$, where $c_m/c_p = 4, \alpha = 2, \beta_1 = 2, \beta_2 = 3$, and

$$a_k = \frac{6k+1}{5k+1}, \qquad b_k = \frac{k}{2k+1}, \qquad k = 0, 1, 2, \ldots.$$

From Table 1, we can see that more PM intervals are needed and the PM intervals at the beginning tend to be larger for the case with nonmaintainable failure modes considered than when the nonmaintainable failure modes are ignored. The PM interval is becoming shorter and shorter except the last one for both cases. Besides, the shortening speed at the beginning is much faster when non-maintainable failure modes are considered than when they are ignored.

For Model 2, solving Eq. (5) with respect to y_k $(k = 1, 2, \ldots, N)$, we have

$$y_k = \left(\frac{\lambda}{\beta_1 + A_k\beta_2}\right)^{1/(\alpha-1)}, \qquad k = 1, 2, \ldots, N. \tag{11}$$

Substituting Eq. (11) into Eq. (6) and solving it with respect to λ, we get

$$\lambda = \left[\frac{c_r + c_p(N-1)}{(1-1/\alpha)c_m E(N)}\right]^{(\alpha-1)/\alpha}, \tag{12}$$

where

$$E(N) = \sum_{k=1}^{N-1} \left(1 - \frac{\beta_1 + A_{k+1}\beta_2}{\beta_1 + A_k\beta_2} b_k^\alpha\right) (\beta_1 + A_k\beta_2)^{-1/(\alpha-1)} + (\beta_1 + A_N\beta_2)^{-1/(\alpha-1)}.$$

Then expression (7) becomes

$$\left[\frac{c_r + c_p(N-1)}{(1-1/\alpha)c_m}\right]^{(\alpha-1)/\alpha} \frac{[E(N)]^{1/\alpha}}{F(N)}, \tag{13}$$

Table 1. Optimal PM schedules for Weibull case in Model 1.

	With Nonmaintainable Failure Modes					Without Nonmaintainable Failure Modes				
c_r/c_p	2	5	10	20	50	2	5	10	20	50
N^*	1	3	6	9	13	1	3	5	7	11
x_1	0.447	0.485	0.609	0.775	1.100	0.447	0.504	0.648	0.838	1.207
x_2		0.262	0.329	0.419	0.595		0.249	0.321	0.415	0.597
x_3		0.350	0.258	0.328	0.466		0.310	0.234	0.303	0.436
x_4			0.214	0.272	0.386			0.183	0.237	0.341
x_5			0.180	0.229	0.326			0.267	0.191	0.274
x_6			0.281	0.194	0.276				0.155	0.224
x_7				0.165	0.235				0.238	0.184
x_8				0.140	0.199					0.151
x_9				0.224	0.169					0.125
x_{10}					0.143					0.104
x_{11}					0.120					0.164
x_{12}					0.101					
x_{13}					0.164					

where

$$F(N) = \sum_{k=1}^{N-1} (1-b_k)(\beta_1 + A_k\beta_2)^{-1/(\alpha-1)} + (\beta_1 + A_N\beta_2)^{-1/(\alpha-1)}.$$

Minimization of the function in (13) is equivalent to minimization of function

$$Q(N) = \frac{[c_r + c_p(N-1)]^{(\alpha-1)/\alpha}[E(N)]^{1/\alpha}}{F(N)}.$$

Inequalities $Q(N+1) \geq Q(N)$ and $Q(N) < Q(N-1)$ imply that

$$W(N) \geq \frac{c_r}{c_p} \quad \text{and} \quad W(N-1) < \frac{c_r}{c_p}, \tag{14}$$

where

$$W(N) = \frac{[E(N+1)]^{1/(\alpha-1)}[F(N)]^{\alpha/(\alpha-1)}}{[E(N)]^{1/(\alpha-1)}[F(N+1)]^{\alpha/(\alpha-1)} - [E(N+1)]^{1/(\alpha-1)}[F(N)]^{\alpha/(\alpha-1)}} - (N-1).$$

Thus, the optimal value N^* must satisfy inequalities (14). The optimal PM intervals are computed from $x_k = y_k - b_{k-1}y_{k-1}$ $(k = 1, 2, \ldots, N)$, where y_k $(k = 1, 2, \ldots, N)$ and λ are given in Eqs. (11) and (12).

Table 2 gives the optimal PM schedules in Model 2 with and without nonmaintainable failure modes considered, respectively, for the same parameters adopted in the example of Model 1. From Table 2, we also observe that more PM intervals are needed and the PM intervals at the beginning are larger for the case with nonmaintainable failure modes considered than when the nonmaintainable failure modes are ignored. The PM interval is becoming shorter and shorter for

Table 2. Optimal PM schedules for Weibull case in Model 2.

	With Nonmaintainable Failure Modes					Without Nonmaintainable Failure Modes				
c_r/c_p	2	5	10	20	50	2	5	10	20	50
N^*	1	4	6	9	13	1	3	5	8	11
x_1	0.447	0.517	0.622	0.766	1.067	0.447	0.553	0.671	0.835	1.180
x_2		0.298	0.358	0.441	0.614		0.290	0.351	0.437	0.618
x_3		0.233	0.281	0.346	0.481		0.211	0.257	0.319	0.451
x_4		0.193	0.233	0.287	0.399			0.201	0.250	0.354
x_5			0.196	0.242	0.337			0.162	0.202	0.285
x_6			0.167	0.205	0.286				0.165	0.233
x_7				0.174	0.242				0.135	0.191
x_8				0.148	0.206				0.112	0.158
x_9				0.125	0.174					0.130
x_{10}					0.147					0.108
x_{11}					0.124					0.090
x_{12}					0.105					
x_{13}					0.088					

both cases. Besides, the shortening speed at the beginning is much faster when nonmaintainable failure modes are considered than when they are ignored.

4. CONCLUDING REMARKS

In this paper we have presented a general PM policy which assumes that PM only reduces the hazard rate of maintainable failure modes of the system, but does not affect the hazard rate of nonmaintainable failure modes of the system. Two models for obtaining the optimal PM schedules have been studied. Generally, the optimal solutions in the two models are not the same. When the Hybrid PM Model reduces to the Hazard Rate PM Model $(b_k = 0)$, however, it can be shown that the optimal PM schedules obtained from Model 1 and Model 2 are exactly the same. From the examples, we see that the optimal solution to y_k is analytically found in the special Weibull case. For the general case, however, the optimal y_k may not be analytically obtainable. The algorithms in Section 2 may become numerical procedures and standard numerical approaches for solving an equation are required in Step 1 and Step 2.

From Table 1, we can see that x_k decreases for all k values except for the last one. This coincides with Nakagawa's [15, p. 297] comment: "it would be reasonable to do frequent PM with age, but it would be better to do the last PM as late as possible because the system should be replaced at next PM." From Table 2, however, we observe that x_k decreases for all k values. Therefore, we may conclude that Model 1 would be more practical than Model 2.

In Model 2, the hazard rate at each PM remains the same (equals λ). The predetermined level λ is selected to minimize the mean cost rate. An alternative and reasonable way is to determine λ to meet practical requirement of the system reliability. That is, λ is set to be a specific level λ_0 (λ_0 is known). Then y_k $(k = 1, 2, \ldots, N)$ can be computed from Eq. (5). The mean cost rate C is a function only of N. Thus, we can find N which minimizes C and finally obtain the optimal PM schedule.

Some issues deserving further research include how to separate failure modes into maintainable and nonmaintainable failure modes, how to use maintenance and failure database to obtain life distribution parameters and PM adjustment factors, and when the models presented in this paper and other literatures can be applied effectively.

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