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Optimal replacement policy for a multistate repairable system

YL Zhang[1], RCM Yam[2] and MJ Zuo[3]*

[1]*Southeast University, China;* [2]*City University of Hong Kong, China; and* [3]*University of Alberta, Canada*

In this paper, a deteriorating simple repairable system with $k+1$ states, including k failure states and one working state, is studied. The system after repair is not 'as good as new' and the deterioration of the system is stochastic. Under these assumptions, we study a replacement policy, called policy N, based on the failure number of the system. The objective is to maximize the long-run expected profit per unit time. The explicit expression of the long-run expected profit per unit time is derived and the corresponding optimal solution may be determined analytically or numerically. Furthermore, we prove that the model for the multistate system in this paper forms a general monotone process model which includes the geometric process repair model as a special case. A numerical example is given to illustrate the theoretical results.

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Introduction

The earliest replacement models consider one-component repairable systems with one repairman (called *simple repairable systems*). It is assumed that the system after repair is 'as good as new' and this kind of repair is called a *perfect repair.* The replacement of a piece of equipment with a new one can be considered to be a perfect repair. However, most repairs in practice are not perfect. Consequently, the system after repair cannot be 'as good as new.' Barlow and Hunter[1] introduce a *minimal repair* model in which the repair activities do not change the failure rate of the system (see, eg, Barlow and Proschan[2] or Ascher and Feingold[3] for further details). Brown and Proschan[4] propose an *imperfect repair* model in which the repair is equivalent to a perfect repair with probability p and to a minimal repair with probability $1-p$ $(0 \leqslant p \leqslant 1)$. Other studies on such imperfect repair models are reported by Park,[5] Block *et al*[6] and Kijima.[7]

For a deteriorating repairable system, the successive working time of the system after repair may become shorter and shorter, while the successive repair time of the system may become longer and longer. Ultimately, the system's working time becomes too short and its repair time becomes too long. To model a deteriorating system with this kind of characteristic, Lam[8] introduces a *geometric process repair model.* Using this model, Lam[8] analyses two kinds of replacement policy. One, called policy T, is based on the working age of the system while the other, called policy N, is based on the cumulative number of failures of the system. Explicit expressions of the long-run average cost per unit time under these two kinds of replacement policy are developed. He also proves that policy N is better than policy T. Because the geometric process is a special monotone process, Stadje and Zuckerman[9] introduce a general monotone process repair model to generalize Lam's work. Zhang[10] also generalizes Lam's work using a bivariate replacement policy, called policy (T, N), under which the system is replaced at the working age T or at the time of the Nth failure, whichever occurs first, and shows that the bivariate policy is better than the univariate replacement policies, policy N and policy T. Other research works on the geometric process models include Stadje and Zuckerman,[11] Finkelstein,[12] and Stanley.[13]

In most reported repair/replacement models including the geometric process repair model, it is usually assumed that a system may experience only two possible states: one working state and one failure state. The classical reliability theory is based on this binary assumption that each component or the system is either working perfectly or completely failed. However, in many practical situations, a system may experience more than two possible states. For example, a radio or microwave transmitter may be working with full transmission range, working with degraded transmission range, or completely failed. The health condition of an automobile may be considered excellent, good, or poor. Lately, there has been growing research interest on multi-state reliability theory, models, and optimization algorithms, for example, see Andrzejczak,[14] Huang *et al*[15] and Levitin and Lisnianski.[16,17]

A special type of multi-state system model considers multiple distinct failure modes. For example, a relay circuit

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**Correspondence:* MJ Zuo, Department of Mechanical Engineering, University of Alberta, 4–9 Mechanical Engineering Building, Edmonton, Alberta, T6G 2G8 Canada.*
E-mail: ming.zuo@ualberta.ca

may experience two different failure modes, called dual failure modes, in addition to the working mode. When it is energized and thus required to close, it may fail to do so due to the presence of dust and other insulating media. When it is de-energized and thus required to open, it may fail to do so because the contacts are stuck together due to overheating. A flow control valve may also experience such dual failure modes. Another example is a home security system. It may fail to detect a break-in due to mechanical or electrical circuit failures. It may also create a false alarm due to the presence of a pet. Systems with dual failure modes are studied, for example, by Barlow *et al*,[18] Ben-Dov[19] and Pham and Malon.[20] Pham[21] conducts a reliability analysis of systems with three distinct failure modes. A review of research on systems with dual failure modes is provided by Lesanovsky.[22]

In this paper, a deteriorating simple repairable system with $k+1$ states ($k \geqslant 2$) is studied. Among these states, k of them are failure states and the other one is the working state. The next section includes the assumptions used in the system model. Next, we consider the replacement policy based on the number of failures that the system has experienced and prove that the proposed replacement model for the multistate system forms a general monotone process and that the geometric process repair model is its special case. A numerical example is then given. Finally, some concluding remarks are given.

For ease of reference, we provide the definitions of stochastic ordering and the geometric process as follows.

Definition 1 *Suppose that X and Y are two random variables. X is said to be stochastically greater than Y or Y stochastically less than X if*

$$P(X > \alpha) \geqslant P(Y > \alpha) \quad \textit{for all real } \alpha.$$

Such stochastic ordering of these two random variables is denoted by $X \geqslant_{st} Y$ or $Y \leqslant_{st} X$ (see, eg, Ross[23]). Furthermore, we say that a stochastic process $\{X_n, n = 1, 2, \ldots\}$ is stochastically decreasing if $X_n \geqslant_{st} X_{n+1}$ for all $n = 1, 2, \ldots$ or stochastically increasing if $X_n \leqslant_{st} X_{n+1}$ for all $n = 1, 2, \ldots$.

Definition 2 *Assume that $\{\xi_n, n = 1, 2, \ldots\}$ is a sequence of independent non-negative random variables. If the cumulative distribution function of ξ_n is $F_n(t) = F(a^{n-1}t)$ for $n = 1, 2, \ldots,$ where a is a positive constant, then $\{\xi_n, n = 1, 2, \ldots\}$ is said to form a geometric process. The constant a is called the ratio of the geometric process (see, eg, Lam[8] and Zhang[10] for more details).*

Obviously, if $a > 1$, then $\{\xi_n, n = 1, 2, \ldots\}$ is stochastically decreasing, ie

$$\xi_n \geqslant_{st} \xi_{n+1} \quad n = 1, 2, \ldots$$

If $0 < a < 1$, then $\{\xi_n, n = 1, 2, \ldots\}$ is stochastically increasing, ie

$$\xi_n \leqslant_{st} \xi_{n+1} \quad n = 1, 2, \ldots$$

If $a = 1$, then the geometric process becomes a renewal process.

System description and the replacement model

In this section, we describe the system to be studied and propose a model for optimal replacement decision making.

System description

1. At time 0, a new system is installed. This system will eventually be replaced by a new and identical one.
2. The system may experience $k+1$ different states including one working state and k different failure states. State 0 represents the working state while state i (for $1 \leqslant i \leqslant k$) represents the ith type of failure state of the system. These failure states are mutually exclusive and stochastic.
3. Whenever the system was failed, a repairman starts to repair it right away. The system after repair is not 'as good as new'. We will use S_n, X_n, Y_n, and Z to represent, respectively, the type of the nth failure, the consecutive working time after the $(n-1)$th repair, the repair time after the nth failure, and the time duration required to replace the failed system with a new one when necessary, where $n = 1, 2, \ldots$ (see Figure 1). Let $EX_1 = \lambda > 0, EY_1 = \mu > 0,$ and $EZ = \nu > 0$. S_n may take values in the set $\{0, 1, 2, \ldots, k\}$.
4. Assume that X_n, Y_n, and Z, for $n = 1, 2, \ldots$ are independent random variables.
5. We will use c_1, c_2, c_3, and c_4 to represent, respectively, the working reward of the system per unit of working time, the cost of the system per unit of repair time when failed, the variable replacement cost of the system per unit of time, and the fixed replacement cost of the system.

The replacement model based on the number of failures, N

Because the system after each repair is not 'as good as new', the consecutive working time of the system will become stochastically shorter and shorter and at the same time the

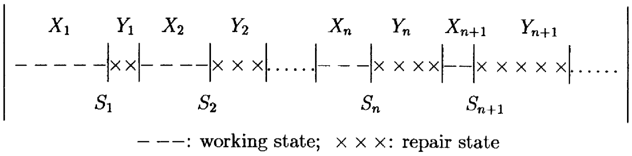

Figure 1 A possible sequence of system state changes.

repair time for each failure will become stochastically longer and longer. Eventually, the system's working time will be too short and the repair time will become too long. As a result, the system will have to be replaced by a new one.

We consider the replacement policy, or policy *N,* which is based on the number of failures of the system. The system will be replaced by a new one when the total number of failures of the system reaches *N*. Our objective is to determine the optimal *N* value, denoted by *N**, such that the long-run expected profit per unit time is maximized.

Let T_1 be the first replacement time point of the system under policy *N*. Let $T_n (n \geqslant 2)$ be the time duration between the $(n-1)$th replacement and the *n*th replacement of the system. Obviously, $\{T_1, T_2, \ldots\}$ forms a renewal process and the interarrival time between two consecutive replacements is called a renewal cycle. Let $C(N)$ be the long-run expected profit per unit time of the system under policy *N*. According to the renewal reward theorem (see Ross[23]), we have

$$C(N) = \frac{\text{the expected profit incurred in a renewal cycle}}{\text{the expected length of the renewal cycle}}$$

$$= \frac{c_1 E(\sum_{n=1}^{N} X_n) - c_2 E(\sum_{n=1}^{N-1} Y_n) - (c_4 + c_3 EZ)}{E(\sum_{n=1}^{N-1} Y_n) + E(\sum_{n=1}^{N} X_n) + EZ} \qquad (1)$$

In the following section we will find a more explicit expression of the objective function $C(N)$. Then, the optimal *N* value will be determined.

Analyses and results

To determine the distribution functions of X_n and Y_n, we first introduce the following probabilities and conditional probabilities:

$$\begin{aligned} &P(X_n \leqslant t, S_n = i | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\qquad \equiv U_i(a_1^{\alpha 1} a_2^{\alpha 2} \cdots a_k^{\alpha k} t) \end{aligned} \qquad (2)$$

$$\begin{aligned} &P(Y_n \leqslant t, S_n = i | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\qquad \equiv V_i(b_1^{\alpha 1} b_2^{\alpha 2} \cdots b_k^{\alpha k} t) \end{aligned} \qquad (3)$$

where

$$i = 1, 2, \ldots, k$$

$$l_1, l_2, \ldots, l_{n-1} \in \{1, 2, \ldots, k\}$$

$$1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$$

$$0 < b_k \leqslant b_{k-1} \leqslant \cdots \leqslant b_1 \leqslant 1$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = n - 1$$

$$n = 1, 2, \ldots$$

In the above definitions, α_i indicates the number of occurrences of failure type *i* among the first $n-1$ failures that the system has experienced for $1 \leqslant i \leqslant k$, a_i represents the impact on the system's lifetime distribution by each occurrence of failure type *i*, and b_i indicates the impact on the duration of repair time by each occurrence of failure type *i*. The conditions $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$ indicate that failure type $i+1$ is more serious than failure type *i* for $i = 1, 2, \ldots, k-1$.

When $n = 1$, we have

$$P(X_1 \leqslant t, S_1 = i) = U_i(t) \qquad i = 1, 2, \ldots, k \qquad (4)$$

$$P(Y_1 \leqslant t, S_1 = i) = V_i(t) \qquad i = 1, 2, \ldots, k \qquad (5)$$

Thus, $U_i(t)$ is the probability that the first failure is of type *i* and the system's lifetime to the first failure is less than or equal to *t* and $V_i(t)$ is the probability that the first failure is of type *i* and the time needed to repair this failure is less than or equal to *t*. Since $\{X_n < \infty\}$ and $\{Y_n < \infty\}$ are both sure events for $n = 1, 2, \ldots,$ we have

$$P(S_1 = i) = P(X_1 < \infty, S_1 = i) = \lim_{t \to \infty} U_i(t) \qquad (6)$$

For $n > 1$, we can write

$$\begin{aligned} &P(S_n = i | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\quad = P(X_n < \infty, S_n = i | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\quad = \lim_{t \to \infty} U_i(a_1^{\alpha 1} a_2^{\alpha 2} \cdots a_k^{\alpha k} t) \end{aligned} \qquad (7)$$

It is easy to verify that, for any $n \geqslant 1$,

$$\lim_{t \to \infty} U_i(t) = \lim_{t \to \infty} U_i(a_1^{\alpha 1} a_2^{\alpha 2} \cdots a_k^{\alpha k} t) \equiv p_i \qquad (8)$$

Then, we have

$$\begin{aligned} P(S_n = i) &= \sum_{l_j, j=1,\ldots,n-1} P(S_n = i | S_1 = l_1, \ldots, S_{n-1} = l_{n-1}) \\ &\quad \times P(S_1 = l_1, \ldots, S_{n-1} = l_{n-1}) \\ &= p_i \sum_{l_j, j=1,\ldots,n-1} P(S_1 = l_1, \ldots, S_{n-1} = l_{n-1}) \\ &= p_i \qquad i = 1, 2, \ldots, k; \quad n = 1, 2, \ldots \end{aligned} \qquad (9)$$

Equation (9) indicates that for any $n \geqslant 1$, the probability of event $(S_n = i)$ is always equal to p_i, where $i = 1, 2, \ldots, k$.

According to Equation (2), we have

$$\begin{aligned} &P(X_n \leqslant t | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\quad = \sum_{i=1}^{k} P(X_n \leqslant t, S_n = i | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\quad = \sum_{i=1}^{k} U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \qquad n = 1, 2, \ldots \end{aligned} \qquad (10)$$

Similarly, according to Equation (3) we have

$$\begin{aligned} &P(Y_n \leqslant t | S_1 = l_1, S_2 = l_2, \ldots, S_{n-1} = l_{n-1}) \\ &\quad = \sum_{i=1}^{k} V_i(b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_k^{\alpha_k} t) \qquad n = 1, 2, \ldots \end{aligned} \qquad (11)$$

Using the properties of the multinomial distribution we can obtain the cumulative distribution function (CDF) of X_n for $n = 1, 2, \ldots$:

$$\begin{aligned} F_n(t) &= P(X_n \leqslant t) \\ &= \sum_{l_j, j=1,\ldots,n-1} P(X_n \leqslant t | S_1 = l_1, \ldots, S_{n-1} = l_{n-1}) \\ &\quad \times P(S_1 = l_1, \ldots, S_{n-1} = l_{n-1}) \\ &= \sum_{l_j, j=1,\ldots,n-1} \sum_{i=1}^{k} U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \\ &\quad \times P(S_1 = l_1, \ldots, S_{n-1} = l_{n-1}) \\ &= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \frac{(n-1)!}{\alpha_1! \alpha_2! \cdots \alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\ &\equiv F(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \qquad n = 1, 2, \ldots \end{aligned} \tag{12}$$

Similarly, we have the CDF of Y_n for $n = 1, 2, \ldots$:

$$\begin{aligned} G_n(t) &= P(Y_n \leqslant t) \\ &= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} = V_i(b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_k^{\alpha_k} t) \frac{(n-1)!}{\alpha_1! \alpha_2! \cdots \alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \\ &\equiv G(b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_k^{\alpha_k} t) \qquad n = 1, 2, \ldots \end{aligned} \tag{13}$$

For such a deteriorating simple repairable system with multistates, we can obtain the expectations of X_n and Y_n $(n = 1, 2, \ldots)$ as stated in Theorem 1. A proof of Theorem 1 is given in the appendix.

Theorem 1

$$EX_n = \lambda \left(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_k}{a_k} \right)^{n-1} \qquad n = 1, 2, \ldots \tag{14}$$

$$EY_n = \mu \left(\frac{p_1}{b_1} + \frac{p_2}{b_2} + \cdots + \frac{p_k}{b_k} \right)^{n-1} \qquad n = 1, 2, \ldots \tag{15}$$

Based on Theorem 1, we have

$$\begin{aligned} E\left(\sum_{n=1}^{N} X_n \right) &= \lambda \sum_{n=1}^{N} \left(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_k}{a_k} \right)^{n-1} \\ &= \lambda \frac{1 - ((p_1/a_1) + (p_2/a_2) + \cdots + (p_k/a_k))^N}{1 - ((p_1/a_1) + (p_2/a_2) + \cdots + (p_k/a_k))} \end{aligned} \tag{16}$$

$$E\left(\sum_{n=1}^{N-1} Y_n \right) = \mu \frac{1 - ((p_1/b_1) + (p_2/b_2) + \cdots + (p_k/b_k))^{N-1}}{1 - ((p_1/b_1) + (p_2/b_2) + \cdots + (p_k/b_k))} \tag{17}$$

For simplicity of reference, we define A_i and B_i as follows:

$$A_i = \frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_i}{a_i} \qquad 1 \leqslant i \leqslant k \tag{18}$$

$$B_i = \frac{p_1}{b_1} + \frac{p_2}{b_2} + \cdots + \frac{p_i}{b_i} \qquad 1 \leqslant i \leqslant k \tag{19}$$

An explicit expression of the objective function given in Equation (1) can be written as

$$C(N) = \frac{c_1 \lambda (1 - A_k^N / 1 - A_k) - c_2 \mu (1 - B_k^{N-1} / 1 - B_k) - (c_4 + c_3 v)}{\mu (1 - B_k^{N-1} / 1 - B_k) + \lambda (1 - A_k^N / 1 - A_k) + v} \tag{20}$$

For the objective function shown in Equation (20), we can determine the optimal value N^*, using either analytical or numerical methods such that $C(N)$ is maximized at N^*.

For the replacement analysis model proposed above, we have the following observations:

1. It can be shown that the repair model used in this paper is a kind of general monotone repair model, as stated in Theorem 2. A proof of Theorem 2 is given in the appendix.

Theorem 2 *For all* $t > 0$,

$$P(X_n > t) \geqslant P(X_{n+1} > t) \qquad n = 1, 2, \ldots \tag{21}$$

$$P(Y_n > t) \leqslant P(Y_{n+1} > t) \qquad n = 1, 2, \ldots \tag{22}$$

In other words, we have

$$X_n \geqslant_{st} X_{n+1} \qquad n = 1, 2, \ldots \tag{23}$$

$$Y_n \leqslant_{st} Y_{n+1} \qquad n = 1, 2, \ldots \tag{24}$$

2. When $a_1 = a_2 = \cdots = a_k \equiv a$ and $b_1 = b_2 = \cdots = b_k \equiv b$, the repair model for the multistate system in this paper reduces to the geometric process repair model. According to Equation (12), when $n = 1$ we have

$$F_1(t) = P(X_1 \leqslant t) = \sum_{i=1}^{k} U_i(t)$$

When $n = 2$ we have

$$\begin{aligned} F_2(t) &= P(X_2 \leqslant t) \\ &= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = 1} U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \frac{1!}{\alpha_1! \alpha_2! \cdots \alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\ &= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = 1} U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \end{aligned}$$

If there is only one failure state, ie $a_1 = a_2 = \cdots = a_k = a, b_1 = b_2 = \cdots = b_k = b$ and $k = 1$ we have

$$F_1(t) = P(X_1 \leqslant t) = U_1(t) \equiv F(t)$$

$$F_2(t) = P(X_2 \leqslant t) = \sum_{\alpha_1=0}^{1} U_1(at) p_1^{\alpha_1} (1 - p_1)^{1-\alpha_1} = U_1(at) \equiv F(at)$$

Generally, we have

$$\begin{aligned} F_n(t) = P(X_n \leqslant t) &= \sum_{\alpha_1=0}^{n-1} U_1(a^{n-1} t) \frac{(n-1)!}{\alpha_1! [(n-1) - \alpha_1]!} \\ &\times p_1^{\alpha_1} (1 - p_1)^{(n-1)-\alpha_1} = U_1(a^{n-1} t) \equiv F(a^{n-1} t) \end{aligned}$$

Similarly, using Equation (13) we have

$$G_n(t) = P(Y_n \leqslant t) = V_1(b^{n-1} t) \equiv G(b^{n-1} t)$$

According to Definition 2, $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$ form, respectively, a stochastic decreasing geometric process and a stochastic increasing geometric process. In other words, the geometric process repair model used by Lam[8] is a special case of the repair model proposed in this paper.

A numerical example

In this section, we provide a hypothetical example to illustrate the use of the results reported in this paper. Consider a system that has one working state and two distinct failure states. We have

$$EX_n = \lambda\left(\frac{p_1}{a_1} + \frac{p_2}{a_2}\right)^{n-1} \qquad n = 1, 2, \ldots$$

$$EY_n = \mu\left(\frac{p_1}{b_1} + \frac{p_2}{b_2}\right)^{n-1} \qquad n = 1, 2, \ldots$$

According to Equation (20), we have

$$C(N) = \frac{c_1\lambda(1 - A_2^N/1 - A_2) - c_2\mu(1 - B_2^{N-1}/1 - B_2) - (c_4 + c_3 v)}{\mu(1 - B_2^{N-1}/1 - B_2) + \lambda(1 - A_2^N/1 - A_2) + v}$$

When $p_1 = 0.495,\ p_2 = 0.505,\ a_1 = 1.0005,\ a_2 = 1.1,\ b_1 = 0.95,\ b_2 = 0.85$, we have

$$A_2 = \frac{p_1}{a_1} + \frac{p_2}{a_2} = 0.954$$

$$B_2 = \frac{p_1}{b_1} + \frac{p_2}{b_2} = 1.115$$

Assuming further $\lambda = 12, \mu = 25, c_1 = 200, c_2 = 5,\ c_4 = 4000, c_3 = 10$, and $v = 24$, then it is easy to find that $N^* = 6$, such that $C(N)$ is maximized at N^* (see Figure 2).

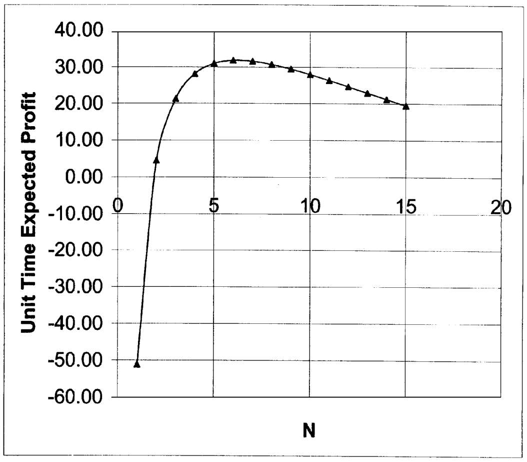

Figure 2 The plot of expected profit rate against *N*.

Conclusions

In this paper, a deteriorating simple repairable system with one working state and $k\,(k \geqslant 2)$ distinct failure states is analysed. A general monotone process model is proposed for replacement decision making. The proposed model includes the geometric process model as a special case. Policy N is used for determination of the optimal replacement time. Future research topics include analyses of systems with multiple components and/or more than one working state.

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Appendix

Proof of Theorem 1

From Equations (12) and (13), when $n = 1$ we have

$$F_1(t) = P(X_1 \leqslant t) = \sum_{i=1}^{k} U_i(t)$$

$$G_1(t) = P(Y_1 \leqslant t) = \sum_{i=1}^{k} V_i(t)$$

As a result, we have

$$\lambda = EX_1 = \int_0^\infty t\mathrm{d}F_1(t)$$

$$\mu = EY_1 = \int_0^\infty t\mathrm{d}G_1(t)$$

$$\begin{aligned} EX_n &= \int_0^\infty t\mathrm{d}F_n(t) \\ &= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1!\alpha_2!\cdots\alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\ &\quad \times \int_0^\infty t\mathrm{d}U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \\ &= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1!\alpha_2!\cdots\alpha_k!} \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}} \int_0^\infty u\mathrm{d}U_i(u) \\ &= \lambda \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1!\alpha_2!\cdots\alpha_k!} \left(\frac{p_1}{a_1}\right)^{\alpha_1} \left(\frac{p_2}{a_2}\right)^{\alpha_2} \cdots \left(\frac{p_k}{a_k}\right)^{\alpha_k} \\ &= \lambda\left(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_k}{a_k}\right)^{n-1} \qquad n = 1, 2, \ldots \end{aligned}$$

Similarly, we can obtain the expression given in Equation (15).

Proof of Theorem 2

According to the distributed function of X_n and the property of multinomial distribution, we have

$$
\begin{aligned}
&P(X_{n+1} \leqslant t) \\
&= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n} U_I(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\
&= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n} \left\{ \sum_{l=1}^{k} \frac{(n-1)!\alpha_l}{\alpha_1! \alpha_2! \cdots \alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \right\} \\
&= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n} \frac{(n-1)! p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} p_1 U_i(a_1 a_1^{\alpha_1 - 1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t)}{(\alpha_1 - 1)! \alpha_2! \cdots \alpha_k!} \\
&\quad + \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n} \frac{(n-1)! p_1^{\alpha_1} p_2^{\alpha_2 - 1} \cdots p_k^{\alpha_k} p_2 U_i(a_2 a_1^{\alpha_1} a_2^{\alpha_2 - 1} \cdots a_K^{\alpha_k} t)}{\alpha_1! (\alpha_2 - 1)! \cdots \alpha_k!} \\
&\quad + \cdots \\
&\quad + \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n} \frac{(n-1)! p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k - 1} p_k U_i(a_k a_1^{\alpha_1} \cdots a_k^{\alpha_k - 1} t)}{\alpha_1! \alpha_2! \cdots \alpha_{k-1} (\alpha_k - 1)!} \\
&= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)! p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \left\{ \sum_{l=1}^{k} p_l U_i(a_l a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \right\}}{\alpha_1! \alpha_2! \cdots \alpha_k!} \\
&\geqslant \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)! p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \left\{ \sum_{l=1}^{k} p_l U_i(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \right\}}{\alpha_1! \alpha_2! \cdots \alpha_k!} \\
&= \sum_{i=1}^{k} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \cdots \alpha_k!} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} U_i(a_1^{\alpha_1} \cdots a_k^{\alpha_k} t) \\
&= P(X_n \leqslant t)
\end{aligned}
$$

Therefore, we have $P(X_n > t) > P(X_{n+1} \geqslant t)$. Similarly, Equation (22) can be obtained.

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